Analysis Qualifying exam
Fall 2005

Answer all questions.

R1 Suppose that f is a bounded function on $[a, b]$ which is Riemann integrable.

a) Prove that f is a Lebesgue measurable function.

b) Must f be a Borel measurable function? Prove your assertion.

R2 Let X be a set with $\mathcal{S}$ a σ-algebra of sets in X, and F a signed measure on $\mathcal{S}$ with $F(S) > -\infty$ for all $S \in \mathcal{S}$. A subset $B \in \mathcal{S}$ is said to be *purely positive* if $F(C) \geq 0$ for all $C \in \mathcal{S}$ with $C \subseteq B$, and *purely negative* if $F(C) \leq 0$ for all $C \in \mathcal{S}$ with $C \subseteq B$. Prove that there exist $P, N \in \mathcal{S}$ with $X = P \cup N$ and $P \cap N = \emptyset$, where P is purely positive and N is purely negative. Hint: Begin by showing that $\beta = \inf\{F(B) : B \text{ is purely negative}\} > -\infty$.

R3 Consider real numbers $a_{n,m}$ for $n = 1, 2, \ldots$ and $m = 1, 2, \ldots$ and assume that the inner and outer sums in the expressions

$$A \ : = \sum_{n=1}^{\infty} \left[\sum_{m=1}^{\infty} a_{n,m} \right]$$

$$B \ : = \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} a_{n,m} \right]$$

are absolutely convergent.

a) Give an example that shows that we may have $A \neq B$.

b) Under what reasonable additional assumption on $a_{n,m}$ can we conclude that $A = B$? Prove your assertion.

R4 Suppose that V is a complex normed space and that $f : V \to \mathbb{C}$ is a linear functional.

a) Prove that if V is finite-dimensional, then f must be bounded.

b) Give an example of an unbounded linear functional on a normed vector space. Prove your assertion.

R5 (see hint below) If $f \in L^1(\mathbb{R})$, consider the Fourier transform

$$\widehat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ix\alpha} dx$$

and the inverse Fourier transform

$$\check{f}(x) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{ix\alpha} d\alpha$$

(you may use any of the alternative standard definitions for the Fourier transform and its inverse).

(a) Let $A(\mathbb{R})$ be the image of the mapping $f \mapsto \widehat{f}$. Does one have that $A(\mathbb{R}) \subseteq L^1(\mathbb{R})$ or $A(\mathbb{R}) \supseteq L^1(\mathbb{R})$? Fully prove your assertions.

(b) Prove that

$$f, g \in L^2(\mathbb{R}) \Longrightarrow (\hat{f}\hat{g})\check{} = f * g.$$

Hint: Let $h(y) = \overline{g(x-y)}$ and observe that $\hat{h}(\alpha) = \overline{\hat{g}(\alpha)} e^{-2\pi i \alpha}$. Then use the fact that $f \mapsto \widehat{f}$ uniquely determines an isometry of $L^2(\mathbb{R})$ onto itself (the Plancherel theorem).

C1 Suppose that $f(z)$ is analytic and non-constant on a connected open set G in the complex plane. Prove that $f(G)$ is an open subset of the complex plane.

C2 Find an explicit conformal mapping from the upper half-plane slit along the vertical segment

$$\{z \in \mathbb{C} : \operatorname{Im} z > 0\} \backslash (0, i]$$

to the unit disk $\{z \in \mathbb{C} : |z| < 1\}$.

C3 Consider the meromorphic function

$$f(z) = \frac{(1-z^2)}{2i(z^2 - (a + \frac{1}{a})z + 1)}, \quad |a| < 1.$$

Find the Laurent series expansion for $f(z)$ valid in a neighborhood of the unit circle $|z| = 1$.

C4 Using the residue calculus, evaluate the integral

$$\int_0^\infty \frac{\log x}{(x^2+1)^2} dx.$$

Hint: Use the positively oriented contour $\Gamma_{r,R}$ $0 < r < 1 < R$ consisting of the line segment $[r, R]$, followed by the arc $\Gamma_{r,R} = \{z = Re^{i\varphi} : 0 \le \varphi \le \pi\}$, the segment $-R, -r]$, and finally completed by the arc

$$\Gamma_r = \left\{z = re^{i\varphi} : 0 \le \varphi \le \pi\right\}.$$

Include a proof of the limiting arguments.

C5 Let $J \subseteq \mathbb{R}$ be a compact interval, and let μ be a measure on the real line whose support lies in J. For $z \in \mathbb{C}\backslash J$, define

$$C_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{z-t}.$$

a) Prove that $C_\mu(z)$ is analytic on $\mathbb{C}\backslash J$.

b) Find a power series expansion for $C_\mu(z)$ at ∞ in terms of the moments $m_k = \int_{\mathbb{R}} t^k d\mu(t)$.

c) Show that the mapping $\mu \mapsto C_\mu(z)$ is one-to-one (Hint: use the Stone-Weierstrass theorem to prove that the moments m_k determine μ).