ANALYSIS QUALIFYING EXAMINATION FALL 2006

THURSDAY SEPT. 21, 2006

INSTRUCTIONS. Each problem is worth 10 points. There are six real analysis problems, numbered 1–6, and six complex analysis problems, numbered 7–12. You are to solve five problems from each section.

> ! **You must indicate which five problems from each section are to be graded.** !

Please note that a complete solution to a single problem will be valued more highly than two half solutions to two problems.

Problem 1. Let $\theta \in [0,1]$ be an irrational number. Let X be the unit circle $\{e^{2\pi it} : t \in [0,1]\}$ endowed with the arclength measure, and let

$$\alpha : X \to X$$

be the rotation by $2\pi\theta$, given by $\alpha(e^{2\pi it}) = e^{2\pi i(t+\theta)}$.

(1) Show that if $f \in L^2(X)$ and $f \circ \alpha = f$, then f is a.e. constant. (*Hint:* consider the Fourier transform of f, viewed as a 1-periodic function on $\mathbb{R}$).
(2) Use part 1 to show that if $Y \subset X$ is a Lebesgue-measurable subset so that $\alpha(Y) = Y$, then either Y has measure zero, or $X \setminus Y$ has measure zero.

Problem 2. Let E, F be two Lebesgue-measurable subsets of $\mathbb{R}$, and let χ_E, χ_F be their respective characteristic functions.

(1) Show that the convolution $\chi_E * \chi_F$ defined by

$$\chi_E * \chi_F(x) = \int_{\mathbb{R}} \chi_E(y)\chi_F(x-y)dy$$

is a continuous function.
(2) Show that

$$n\left(\chi_E * \chi_{[0,1/n]}\right) \to \chi_E$$

as $n \to \infty$ pointwise a.e.

Problem 3. Let $(X, \mathcal{M}, \mu)$ be a measure space, $\mu(X) = 1$. Let $f \in L^\infty(X, \mathcal{M}, \mu)$. Prove that

$$\lim_{p\to\infty} \left(\int |f|^p d\mu\right)^{1/p} = \|f\|_\infty.$$

Problem 4. Assume that f is a continuously differentiable 2π periodic function on $\mathbb{R}$. Show that the Fourier series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) \exp(int), \qquad t \in \mathbb{R}$$

is absolutely convergent (here $\hat{f}(n) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(t)\exp(-int)dt$).

Problem 5. Let ℓ^2 be the space of all square-summable sequences of complex numbers, and let $T : \ell^2 \to \ell^2$ be a linear operator. Let e_n be the sequence

$$e_n = (0\,0\,\cdots\,0\,1\,0\,\cdots),$$

where 1 is in the n-th position. (The vectors e_n form a "standard" orthonormal basis for ℓ^2). Let $a_{nm} = \langle Te_m, e_n \rangle$ be the "matrix coefficients" of T.

(1) Assume that $\displaystyle\sum_{n,m=1}^{\infty} |a_{nm}|^2 < \infty$. Show that T is a bounded operator.

(2) Assume instead that $\sup\{|a_{nm}| : 1 \le n, m < \infty\}$ is finite. Must T be bounded?

Problem 6. Let $\mathbb{D} = \{z : |z| < 1\}$ be the unit disk in the complex plane, endowed with the usual Lebesgue measure. Let $H = L^2(\mathbb{D})$ be the space of square-integrable complex-valued functions on $\mathbb{D}$.

(1) Show that the functions $\{z^n : n \ge 0\}$ are orthogonal in H.

(2) Do the functions $\{\|z^n\|_{L^2(\mathbb{D})}^{-1} z^n : n \ge 0\}$ form an orthonormal basis for H?

Problem 7. Let $f : \mathbb{C} \to \mathbb{C}$ be entire and injective (i.e. univalent). Prove that f is linear: $f(z) = az + b$.

Problem 8. Suppose that f is holomorphic on the open unit disk $\mathbb{D} = \{z : |z| < 1\}$. Suppose also that for $z \in \mathbb{D}$, $\mathrm{Re}(f(z)) > 0$ and that $f(0) = 1$. Prove that for all $z \in \mathbb{D}$,

$$|f(z)| \le \frac{1 + |z|}{1 - |z|}.$$

Problem 9. Evaluate

$$\int_0^{\pi} \frac{d\theta}{2 + \cos\theta}$$

using the residue theorem.

Problem 10. Find an explicit conformal mapping from the slit disk

$$S = \{z : |z| < 1 \ \text{ and } \ z \notin [\tfrac{1}{2}, 1)\}$$

onto the disk $\mathbb{D} = \{z : |z| < 1\}$. (You may express your answer as an explicit composition of explicit maps).

Problem 11. Show that if $\alpha \in \mathbb{C}$ satisfies $0 < |\alpha| < 1$ and $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ then the equation

$$e^z (z - 1)^n = \alpha$$

has exactly n simple roots in the right half-plane $\{z : \mathrm{Re}(z) > 0\}$.

Problem 12. For $a_n = 1 - \dfrac{1}{n^2}$ let

$$f(z) = \prod_{n=1}^{\infty} \frac{a_n - z}{1 - a_n z}.$$

(1) Show that f defines a holomorphic function on $\mathbb{D} = \{z : |z| < 1\}$.

(2) Prove that f does not have an analytic continuation to any larger disk $\{z : |z| < r\}$ where $r > 1$.