ANALYSIS QUALIFYING EXAMINATION

Sunday, April 2, 2006

Instructions: Work any 10 problems. To pass the exam, you must show a satisfactory knowledge of both Real Analysis (Problems 1–6) and Complex Analysis (Problems 7–12). All problems are worth ten points; parts of a problem do not carry equal weight. You need to tell us which 10 problems you want us to grade.
Great emphasis will be placed on your attention to detail.

Problem 1. Let μ and μ_n, $n \in \mathbb{N}$, be finite positive Borel measures on $\mathbb{R}$ such that

$$\int f(x)\mu_n(\mathrm{d}x) \underset{n\to\infty}{\longrightarrow} \int f(x)\mu(\mathrm{d}x)$$

for all continuous funtions f with compact support.

(1) Show that then for each compact set $K \subset \mathbb{R}$,

$$\limsup_{n\to\infty} \mu_n(K) \le \mu(K).$$

(2) Give an example that shows $\mu(\mathbb{R}) < \limsup_{n\to\infty} \mu_n(\mathbb{R})$ is possible.

Problem 2. Let f_n be a sequence of $L^1(\mathbb{R})$ functions such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)g(x)\,dx = g(0),$$

for each $g \in C_0(\mathbb{R})$ that is, continuous functions vanishing at infinity. Show that $\|f_n\|_{L^1}$ is uniformly bounded but that f_n is not Cauchy in L^1.

Problem 3. Consider the space $L^\infty([0,1],\lambda)$ where λ denotes the Lebesgue measure. Let

$$d(f,g) = \inf_{\epsilon>0}\left[\epsilon + \lambda(\{x : |f(x)-g(x)| > \epsilon\})\right]$$

Prove that d is a metric and that $f_n \to f$ if and only if $f_n \to f$ in measure.
Recall that $f_n \to f$ in measure iff $\lambda(\{x : |f_n(x) - f(x)| > \delta\}) \to 0$ for all $\delta > 0$.

Problem 4. Prove (one direction of) the Ascoli-Arzelà Theorem: Suppose $f_n\colon [0,1] \to \mathbb{R}$ is a sequence of functions such that

(1) $\exists M :\ \forall x \in [0,1],\ \forall n \ge 1,\ |f_n(x)| \le M$

(2) $\forall \epsilon > 0,\ \exists \delta > 0\ :\ \forall n \ge 1,\ |x-y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon$

Then (f_n) has a subsequence that converges uniformly.

Problem 5. Suppose $f \in L^p(\mathbb{R}^n, \mathrm{d}x)$ and $g \in L^q(\mathbb{R}^n, \mathrm{d}x)$ where

$$\tfrac{1}{p} + \tfrac{1}{q} = 1$$

and $1 < p < \infty$. Show that

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\mathrm{d}y$$

is a continuous function with

$$\lim_{|x|\to\infty} (f*g)(x) = 0.$$

Problem 6. Let $\mathcal{H}^2(\mathbb{D})$ denote the space of power series $f(z) = \sum_{n\geq 0} a_n z^n$ where $a_n \in \mathbb{C}$ form an ℓ^2-sequence. These can be regarded as analytic functions on the open unit disc $\mathbb{D}$. Note that $\mathcal{H}^2(\mathbb{D})$ is a complex Hilbert space with the norm $\|f\|^2 = \sum_{n\geq 0} |a_n|^2$.

(1) Show that $L(f) := \int_{-1}^{1} f(x)\,\mathrm{d}x$ defines a bounded linear functional on $\mathcal{H}^2(\mathbb{D})$.

(2) Find $g \in \mathcal{H}^2(\mathbb{D})$ so that $L(f) = \langle f, g\rangle$.

Problem 7. Show that every non-negative harmonic function on $\mathbb{R}^2$ is constant.

Problem 8. Compute the limit

$$\lim_{R\to\infty} \int_0^R \mathrm{e}^{\mathrm{i}x^2}\,\mathrm{d}x.$$

You may use the fact that

$$\lim_{R\to\infty} \int_0^R \mathrm{e}^{-x^2}\,\mathrm{d}x = \tfrac{1}{2}\sqrt{\pi}.$$

without proof.

Problem 9. Show that any analytic function $f\colon \mathbb{C} \to \mathbb{C}$ that obeys

$$|f(z)| = 1 \text{ for all } z \in \mathbb{R}$$

can be written as $f(z) = e^{g(z)}$ for some analytic function $g(z)$.
Hint: Prove an analogue of the Schwarz reflection principle.

Problem 10. Prove that

$$\frac{\pi^2}{\tan^2(\pi z)} = a + \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}.$$

for some constant $a \in \mathbb{C}$.

Problem 11. Let $\epsilon = \frac{1}{100}$ and let $\mathcal{U}$ be an ϵ-neighborhood of the spiral $\{\theta e^{\mathrm{i}\theta} \colon 0 \leq \theta \leq 4\pi\}$ and let $\mathcal{O}$ be an ϵ-neighborhood of the spiral $\{2\theta e^{\mathrm{i}\theta} \colon 0 \leq \theta \leq 2\pi\}$. Let $f\colon \mathcal{U} \to \mathbb{C}$ and $g\colon \mathcal{O} \to \mathbb{C}$ be the corresponding analytic continuations of

$$z \mapsto \log\left(\frac{\cos(z)}{1-z^2}\right)$$

from the ϵ-neighborhood of the origin on the real axis such that $f(0) = 0 = g(0)$. Find the imaginary part of $f(4\pi) - g(4\pi)$.

Problem 12. Prove that the infinite product

$$\prod_{n=0}^{\infty} \left(1 + z^{2^n}\right)$$

converges and equals $(1-z)^{-1}$ for all z in the open unit disc.