ANALYSIS QUALIFYING EXAM FALL 2007

Instructions: Do 10 of the 13 problems, at least three from problems 9-13. All problems are worth the same amount.

1 Let f: $R \to R$ be a function of class C^1, periodic of period 2π, and satisfying $\int_{-\pi}^{+\pi} f(t)\, dt = 0$. Prove that $\int_{-\pi}^{+\pi} |f(t)|^2\, dt \le \int_{-\pi}^{+\pi} |f'(t)|^2\, dt$. Find all such functions for which equality holds.

2 Is there a closed uncountable subset of R which contains no rational numbers? Prove your answer.

3(a) Prove that a complete normed vector space(a Banach space) is either finite-dimensional or has uncountable dimension in the vector space sense, i.e. , it is not generated as finite linear combinations of elements of some countable subset.
(b) Use part (a) to give an example of a vector space that cannot be given the structure of a Banach space, that is, it is not complete in any norm.

4 Prove that not every subset of [0,1] is Lebesgue measurable

5 (a) Prove that if a_n is a decreasing sequence of positive numbers converging to 0, then the series $\Sigma\, (-1)^n a_n$ is convergent.
(b) Prove that if $f:[0, \infty) \to (0, \infty)$ is decreasing and $\lim_{x\to+\infty} f(t) = 0$, then the improper integral $\int_0^{+\infty} f(x) \cos x\, dx$ is convergent.

6 Derive, proving the validity carefully, some series or product expansion(your choice) that yields a method of calculating π.

7 Suppose H is a separable Hilbert space.
(a) Prove: If $T:H\to H$ is a linear mapping such that $\|I - T\| < 1$,where I is the identity map of H to itself, then T is invertible.
(b) Suppose e_n , n=1,2,3,…,is a complete, orthonormal set in H(a Hilbert space basis) . Suppose also that f_n ,n=1,2,3,…is an orthonormal set in H such that $\sum \|e_n - f_n\|^2 < 1$.
Prove that f_n is a complete orthonormal set in H.

8 Let $f:R\to R$ be a continuous function that is periodic with period 2π. Show that if all the Fourier coefficients of f (for $[-\pi, +\pi]$) are 0, then f is identically 0.

9 Suppose that h: D-{(0,0)}$\rightarrow$R is a harmonic function, where D is the unit disc in the complex plane.
(a) Show that there is exactly one real number A such that h(z)- A ln$|z|$ is the real part of a holomorphic function on $D-\{(0,0)\}$. (Hint: Consider a candidate for a harmonic conjugate constructed by contour integration.)
(b) Use part (a) to show that if h is bounded then h extends to be a harmonic function on D.

10 Let U be a bounded connected open set in C. Prove that if K is a compact subset of U, then there is a constant C_K such that for every point z in K and every holomorphic L^2 function f on U, $|f(z)| \leq C_K \left(\int_U |f|^2 \right)^{\frac{1}{2}}$.

11 Let U be a bounded connected open set in C. Suppose that 0 belongs to U and that F:U$\rightarrow$U is a holomorphic function with F(0)=0 and with F'(0)=1. Prove that F is the identity map. [Suggestion: Consider the power series of F composed with itself many times].

12 Find a conformal map to the unit disc of the half disc $\{z : |z| < 1, \mathrm{Re}\, z > 0\}$. You may write your answer as a composition of simpler conformal maps.

13 Suppose R is a positive number and U={ z : $|z| < R$}

(a) Show that for each holomorphic function f on U, there is a power series $\sum_{n=o}^{+\infty} a_n z^n$ which converges at each point z in U to f(z).

(b) Show that there is only one such power series.