ANALYSIS QUALIFYING EXAMINATION

March 31, 2007

Instructions: Solve any 10 problems and therefore at least 4 from Problems 1–6 and at least 4 from Problems 7–12. All problems are worth the same amount. Turn in only the 10 problems you want us to grade.

Notation: Throughout, $L^p(A)$ denotes the standard L^p space defined relative to the Lebesgue measure on A and $\|f\|_p$ denotes the corresponding norm.

Problem 1. Let $f_n\colon \mathbb{R} \to \mathbb{R}$ be non-negative integrable functions with $\|f_n\|_1 = 1$. Suppose $f_n \to f$ pointwise a.e. with $\|f\|_1 = 1$. Show that

$$\int_A f_n(x)\mathrm{d}x \underset{n\to\infty}{\longrightarrow} \int_A f(x)\mathrm{d}x$$

uniformly in the choice of Borel set $A \subset \mathbb{R}$. *Hint*: First prove that $f_n \to f$ in L^1.

Problem 2. Consider the function $f\colon (0,\infty)\times(0,\infty) \to \mathbb{R}$ defined by

$$f(x,y) = \sum_{n=0}^{\infty} \frac{x}{x^2 + yn^2}$$

Show that the limit $g(y) := \lim_{x\to\infty} f(x,y)$ exists for all $y > 0$ and compute $g(y)$.

Problem 3. Let $f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)\mathrm{d}y$ denote the convolution of f and g. Fix $g \in L^1(\mathbb{R})$. Do the following:

(1) Show that $A_g(f) := f * g$ is a bounded operator $L^1(\mathbb{R}) \to L^1(\mathbb{R})$.
(2) Suppose in addition $g \geq 0$. Find the corresponding norm $\|A_g\|$.
(3) Show that the only $f \in L^1(\mathbb{R})$ for which $f * f = f$ is $f = 0$.

Problem 4. Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and let $T\colon L^2([0,1]) \to L^2([0,1])$ be defined by

$$(Tf)(x) = f(x + \alpha \bmod 1).$$

Denote $S_n f = f + Tf + T^2 f + \cdots + T^{n-1} f$. Do the following:

(1) For any $f \in L^2([0,1])$, prove that $\frac{1}{n}S_n f$ converges in L^2. Identify the limit.
(2) Suppose $f\colon [0,1] \to \mathbb{R}$ is continuous with $f(1) = f(0)$. Show that the convergence in (1) is uniform.

Problem 5. Let $A_n(f) = \frac{1}{n}\int_0^n f(x)\mathrm{d}x$. Show that there exists a continuous linear functional $A\colon L^\infty(\mathbb{R}_+) \to \mathbb{R}$ such that

$$A(f) = \lim_{n\to\infty} A_n(f)$$

whenever the limit exists. Here $\mathbb{R}_+ = (0, \infty)$.

Problem 6. Let X be a Banach space and let $A\colon X \to X$ be a linear map. Define

$$\varrho(A) = \{\lambda \in \mathbb{C}\colon (\lambda - A) \text{ maps } X \text{ onto } X\}$$

Show that $\varrho(A)$ is an open subset of $\mathbb{C}$.

Problem 7. Use contour integration to evaluate the integral

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{\tan\theta + \mathrm{i}a}$$

where $a \in (0, \infty)$.

Problem 8. Determine the number of zeros of the polynomial

$$p(z) = z^4 + z^3 + 4z^2 + 2z + 3$$

in the right half-plane $\{z\colon \mathrm{Re} z > 0\}$.

Problem 9. Let $f(z)$ be analytic for $0 < |z| < 1$. Suppose there are $C > 0$ and $m \geq 1$ such that

$$\left|f^{(m)}(z)\right| \leq \frac{C}{|z|^m}, \qquad 0 < |z| < 1.$$

Show that $f(z)$ has a removable singularity at $z = 0$.

Problem 10. Let $J = \{\mathrm{i}y\colon 1 \leq y < \infty\}$ and let $\mathbb{H} = \{z\colon \mathrm{Im} z > 0\}$ be the open upper half plane. Consider the domain $D = \mathbb{H} \setminus J$. Find a bounded harmonic function $u\colon D \to \mathbb{R}$ such that $u(x + \mathrm{i}y) \to 0$ as $y \downarrow 0$ and $u(z) \to 1$ as $z \to J$. It is fine to represent to solution in terms of a composition of conformal maps.

Problem 11. Prove that a meromorphic function $f(z)$ in the extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ is the sum of the principal parts at its poles.

Problem 12. Let D be a domain (connected open set) in $\mathbb{C}$ and let (u_n) be a sequence of harmonic functions $u_n\colon D \to (0, \infty)$. Show that if $u_n(z_0) \to 0$ for some $z_0 \in D$, then $u_n \to 0$ uniformly on compact subsets of D.