Analysis Qualifying Examination

Wednesday, September 17, 2008
9am-1pm

Instructions: Work any 10 problems. To pass the exam, you must show a satisfactory knowledge of both Real Analysis (Problems 1-6) and Complex Analysis (Problems7-12). All problems are worth ten points; parts of a problem may not carry equal weight. You need to tell us which 10 problems you want us to grade. Great emphasis will be placed on your attention to detail.

1. Fix $1 \le p < \infty$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Lebesgue measurable functions $f_n : [0,1] \to \mathbb{C}$. Suppose there exists f in $L^p([0,1])$ so that $f_n \to f$ in the L^p sense, that is ,

$$\int_0^1 |f_n(x) - f(x)|^p \, dx \to 0 .$$

(a) Show that $f_n \to f$ in measure, that is, $\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \ge \varepsilon\}) = 0$ for all $\varepsilon > 0$.

(Here μ=Lebesgue measure.)

(b) Show that there is a subsequence f_{n_k} such that $f_{n_k}(x) \to f(x)$ almost everywhere.

2. Is every vector space isomorphic as a vector space to some Banach space? Prove your answer. (Banach space=complete normed vector space, as usual).

3. Prove: If $f : [0,1] \to \mathbb{R}$ is an arbitrary function, not necessarily measurable, then the set of points at which f is continuous is a Lebesgue-measurable set.
(Suggestion: for $x \in \mathbb{R}$, $\delta > 0$, set $S_x(\delta) = \sup\{|f(x_1) - f(x_2)| : |x_1 - x| < \delta, |x_2 - x| < \delta\}$.
Consider the function of $x \in \mathbb{R}$, $\omega(x) = \lim_{\delta \to 0^+} S_x(\delta)$. Caution: it might be $+\infty$ for some x values.)

4. Let X be a subset of $\ell^2(\mathbb{Z})$. Show that X is precompact (i.e., has compact closure) in the $\ell^2(\mathbb{Z})$ topology if and only if X is bounded and

$$\forall \varepsilon > 0, \exists\, N \ge 1 \text{ such that } \forall x \in X, \sum_{|n| \ge N} |x_n|^2 < \varepsilon$$

5. Let $d\mu$ be a finite positive Borel measure on $[0,2\pi]$ and suppose

$$\limsup_{n\to\pm\infty}\left|\int e^{in\theta}d\mu(\theta)\right|=0.$$

Show that for any $f\in L^1(d\mu)$,

$$\limsup_{n\to\pm\infty}\left|\int e^{in\theta}f(\theta)d\mu(\theta)\right|=0.$$

6. Define for each n= 1,2,3.., the Cantor-like set C_n as $[0,1]$ with its central open interval of length $\frac{1}{2^n}\cdot\frac{1}{3}$ removed, then with the two central open intervals of length $\frac{1}{2^n}\cdot\frac{1}{3^2}$ removed from the remaining two closed intervals and so on(at the j^{th} stage, 2^{j-1} intervals of length $\frac{1}{2^n}\cdot\frac{1}{3^j}$ are removed), continuing with $j= 1,2,3\ldots$

(a) With μ= Lebesgue measure, show that $\mu([0,1]-\bigcup_{n=1}^{+\infty}C_n)=0$

(b) Show that if E is a subset of $[0,1]$ which is not Lebesgue measurable (you may assume such an E exists without proof), then for some $n\geq 1$, $E\cap C_n$ fails to be Lebesgue measurable.

(c) Use part (b) to show that there is a continuous, strictly increasing function $f:\mathbb{R}\to\mathbb{R}$ with $f(\mathbb{R})=\mathbb{R}$ and a Lebesgue measurable set $A\subset\mathbb{R}$ such that $f(A)$ is not Lebesgue measurable.

7. If $h:\{z\in\mathbb{C}:1<|z|<2\}\to\mathbb{R}$ is a continuous function, set for $1<r<2$:

$$M_h(r)=\frac{1}{2\pi}\int_0^{2\pi}h(re^{i\theta})d\theta$$

(a) Show that if $h=\operatorname{Re}F$, $F:\{z\in\mathbb{C}:1<|z|<2\}\to\mathbb{C}$ holomorphic, then $M_h(r)$ is constant on $\{r: 1<r<2\}$.

(b) Show that if h is a real-valued harmonic function on $\{z\in\mathbb{C}:1<|z|<2\}$, then there are constants $c_1, c_2\in\mathbb{R}$ such that $M_h(r)=c_1\ln r+c_2$ for all $r\in(1,2)$.

8. Suppose f: $\{z \in \mathbb{C} : 0 < |z| < 1\} \to \mathbb{C}$ is a holomorphic function with $\int_U |f|^2 < +\infty$ where $U = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and the integral is the usual $\mathbb{R}^2$ – area integral. Prove that f has a removable singularity at z=0.

9. Let $D := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in the complex plane and let $H := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ denote the upper half plane .

(a) Explicitly describe all conformal mappings g from H onto D that obey $g(i)=0$.

(b) Suppose f: D→H has $f(0) = i$, f holomorphic. Show that $\operatorname{Im} f(x) \geq \frac{1-x}{1+x}$ for all $x \in (0,1)$.

10. Suppose U is a bounded connected open set in $\mathbb{C}$ and $z_0 \in U$.

Let $F = \{f : U \to D, f \text{ holomorphic}, f(z_0) = 0\}$ where $D=\{z \in \mathbb{C}: |z|<1\}$.

(a) Show that if K is a compact subset of U, then there is a constant $M_K>0$ such that $|f'(z)| \leq M_K$ for all $z \in K$, $f \in F$

(b) Use part (a) to show that if $\{f_n : f_n \in F\}$ is a sequence in F, then there is a subsequence $\{f_{n_j}\}$ which converges uniformly on every compact subset of U to a function $f_0 \in F$.

(Note: Part of this is to show $f_0(U) \subset D$.)

11. Let $D := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in the complex plane and let $\overline{D}$ denote its closure. Suppose f: D→$\mathbb{C}$ is continuous on $\overline{D}$ and analytic (holomorphic) in its interior. Show that if f takes only real values on $\partial\overline{D} := \{z: |z| = 1\}$, then f must be constant.

12. Evaluate $\int_0^{\pi} \frac{d\theta}{a^2 + \sin^2\theta}$ for all real numbers a>0.