Instructions: Attempt ten of the thirteen questions, including at least three from Q9-13. Each question is worth 10 points.

Q1. Let $S^1 := \{z \in \mathbf{C} : |z| = 1\}$ denote the unit circle. Show that there exists a measurable function $f : S^1 \to S^1$ whose Fourier coefficients $\hat{f}(n) := \frac{1}{2\pi}\int_0^{2\pi} f(e^{2\pi i\theta})e^{-2\pi i\theta n}\, d\theta$ are non-zero for every integer $n \in \mathbf{Z}$. (Hint: use the Baire category theorem.)

Q2. Let $\mathbf{R}/\mathbf{Z}$ be the unit circle with the usual Lebesgue measure. For each $n = 1, 2, 3, \ldots$, let $K_n : \mathbf{R}/\mathbf{Z} \to \mathbf{R}^+$ be a non-negative integrable function such that $\int_{\mathbf{R}/\mathbf{Z}} K_n(t)\, dt = 1$ and $\lim_{n\to\infty} \int_{\varepsilon \le |t| \le 1/2} K_n(t)\, dt = 0$ for every $0 < \varepsilon < 1/2$, where we identify $\mathbf{R}/\mathbf{Z}$ with $(-1/2, 1/2]$ in the usual manner. (Such a sequence of K_n are known as *approximations to the identity.*) Let $f : \mathbf{R}/\mathbf{Z} \to \mathbf{R}$ be continuous, and define the convolutions $f * K_n : \mathbf{R}/\mathbf{Z} \to \mathbf{R}$ by

$$f * K_n(x) := \int_{\mathbf{R}/\mathbf{Z}} f(x-t)K_n(t)\, dt.$$

Show that $f * K_n$ converges uniformly to f.

Q3. Let X be a compact metric space.

- (a) Show that X is separable (i.e. it has a countable dense subset).
- (b) Show that X is second countable (i.e. there exists a countable base for the topology).
- (c) Show that $C(X)$ (the space of continuous functions $f : X \to \mathbf{R}$ with the uniform topology) is separable. (Hint: use part (b), Urysohn's lemma and the Stone-Weierstrass theorem.)

Q4. Let $f, g \in L^2(\mathbf{R})$ be two square-integrable functions on $\mathbf{R}$ (with the usual Lebesgue measure). Show that the convolution

$$f * g(x) := \int_{\mathbf{R}} f(y)g(x-y)\, dy$$

of f and g is a bounded continuous function on $\mathbf{R}$.

Q5. Let H be a Hilbert space, and let $T : H \to H$ be a bounded linear operator on H.

- Show that if the operator norm $\|T\|$ of T is strictly less than 1, then the operator $1 - T$ is invertible.
- Let $\sigma(T)$ denote the set of all complex numbers z such that $T - zI$ is not invertible. (This set is known as the *spectrum* of T.) Show that $\sigma(T)$ is a compact subset of $\mathbf{C}$.

Q6. Let μ_n be a sequence of Borel probability measures on $[0,1]$, thus each μ_n is a non-negative finite measure on the Borel σ-algebra of $[0,1]$ (the σ-algebra generated by the open sets in $[0,1]$) with $\mu_n([0,1]) = 1$. Show that there exists a subsequence μ_{n_j}, as well as another Borel probability measure μ, such that $\lim_{j\to\infty} \int_{[0,1]} f(x)\, d\mu_{n_j}(x) = \int_{[0,1]} f(x)\, d\mu(x)$ for all continuous functions $f : [0,1] \to \mathbf{R}$. (Hint: use the Riesz representation theorem and Q3.)

Q7. Let $u : \mathbf{R}^2 \to \mathbf{R}$ be a bounded smooth function, and suppose that the Laplacian $\Delta u(x,y) := \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y)$ of u is rotationally symmetric, which means that $\Delta u(R_\theta(x,y)) = \Delta u(x,y)$ for any rotation $R_\theta : (x,y) \mapsto (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$. Show that u is also rotationally symmetric. (Hint: You may use without proof the fact that the Laplacian Δ commutes with all rotations R_θ.)

Q8. Let H be a real Hilbert space, let K be a closed non-empty subset of H, and let v be a point in H. Show that there exists a unique $w \in K$ which minimizes the distance to v in the sense that $\|v - w\| < \|v - w'\|$ for all $w' \in K \backslash \{w\}$. (Hint: you may find the parallelogram law $\frac{\|a\|^2+\|b\|^2}{2} = \|\frac{a+b}{2}\|^2 + \|\frac{a-b}{2}\|^2$ to be useful.)

Q9. Show using the residue theorem that

$$\int_0^\infty \frac{\log^2 x}{1+x^2} = \frac{\pi^3}{8}.$$

Q10. Let the power series $f(z) = \sum_{n=0}^\infty a_n z^n$ have radius of convergence $r > 0$. For each ρ with $0 < \rho < r$ let $M_f(\rho) := \sup\{|f(z)|; |z| = \rho\}$. Show that the following holds for each such ρ:

$$\sum_{n=0}^\infty |a_n|^2 \rho^{2n} \leq M_f(\rho)^2.$$

Q11. Let $\mathbb{D} := \{z; |z| < 1\}$ be the unit disc. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map having 2 unequal fixed points $a, b \in \mathbb{D}$. Show that $f(z) = z$ for all $z \in D$. (Hint: use Schwartz's lemma.)

Q12. Consider the annulus $A := \{z \in \mathbf{C} : r < |z| < R\}$, where $0 < r < R$. Show that the function $f(z) := 1/z$ cannot be uniformly approximated in A by complex polynomials.

Q13. Let $\Omega \subset \mathbf{C}$ be an open set containing the closed unit disk $\overline{\mathbb{D}} := \{z \in \mathbf{C} : |z| \leq 1\}$, and let $f_n : \Omega \to \mathbf{C}$ be a sequence of holomorphic functions on Ω which converge uniformly on compact subsets of Ω to a limit $f : \Omega \to \mathbf{C}$. Suppose that $|f(z)| \neq 0$ whenever $|z| = 1$. Show that there is a positive integer N such that for $n \geq N$, the functions f_n and f have the same number of zeros in the unit disk $\mathbb{D} := \{z \in \mathbf{C} : |z| < 1\}$.