ANALYSIS QUALIFYING EXAM, FALL 2009

Instructions: Work any 10 problems and therefore at least 4 from Problems 1-6 and at least 4 from Problems 7-12. All problems are worth ten points. Full credit on one problem will be better than part credit on two problems. If you attempt more than 10 problems, indicate which 10 are to be graded.

1: Find a non-empty closed set in the Hilbert space $L^2([0,1])$ that does not contain an element of smallest norm. Prove your assertion.

2: Let v be a trigonometric polynomial in two variables, that is,

$$v(x,y) = \sum_{n,m\in\mathbb{Z}} a_{n,m} e^{2\pi i(nx+my)} ,$$

with only finitely many non-zero coefficients $a_{n,m}$. If

$$u = v - \Delta v$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplacian, prove that

$$\|v\|_{L^\infty([0,1]\times[0,1])} \le C\|u\|_{L^2([0,1]\times[0,1])}$$

for some constant C independent of v.

3: Let $f : [0,1] \to \mathbb{R}$ be continuous with

$$\min_{0\le x\le 1} f(x) = 0 .$$

Assume that for all $0 \le a < b \le 1$ we have

$$\int_a^b \left[f(x) - \min_{a\le x\le b} f(y)\right] dx \le \tfrac{1}{2}|b-a|$$

a) Prove that for all $\lambda \ge 0$:

$$|\{x : f(x) > \lambda + 1\}| \le \tfrac{1}{2}|\{x : f(x) > \lambda\}|$$

Here $|S|$ denotes the Lebesgue measure the set S.
b) Prove that for all $1 \le c < 2$,

$$\int_0^1 c^{f(x)}\, dx \le \frac{100}{2-c}$$

[The constant 100 is not optimal, but is certainly large enough.]

4: Prove the following variant of the Lebesgue Differentiation theorem: Let μ be a finite Borel measure on $\mathbb{R}$, singular with respect to Lebesgue measure. Then for Lebesgue-almost every $x \in \mathbb{R}$,

$$\lim_{\epsilon \to 0} \frac{\mu([x-\epsilon, x+\epsilon])}{2\epsilon} = 0$$

5: Construct a Borel subset E of the real line $\mathbb{R}$ such that for all intervals $[a,b]$ we have

$$0 < m(E \cap [a,b]) < |b-a|$$

where m denotes Lebesgue measure.

6: The Poisson kernel for $0 \le \rho < 1$ is the 2π periodic function on $\mathbb{R}$ defined by

$$P_\rho(\theta) = \operatorname{Re}\left(\frac{1+\rho e^{i\theta}}{1-\rho e^{i\theta}}\right)$$

For functions h continuous on and harmonic inside the closed disc of radius R about the origin one has (you need not prove this)

$$h(re^{i\eta}) = \frac{1}{2\pi}\int_0^{2\pi} P_{r/R}(\eta-\theta) h(Re^{i\theta})\, d\theta$$

Assume that h is harmonic and positive on the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. Prove that there exists a positive Borel measure μ on $[0, 2\pi]$ such that for all $re^{i\theta} \in D$ one has

$$h(re^{i\eta}) = \int_0^{2\pi} P_r(\eta-\theta)\, d\mu(\theta)$$

7: a) Define *unitary operator* on a complex Hilbert space.
b) Let S be a unitary operator on a complex Hilbert space. Using your definition, prove that for every complex number $|\lambda| < 1$ the operator $S - \lambda I$ is invertible. Here I denotes the identity operator.
c) For a fixed vector v in the Hilbert space and all $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$, we define

$$h(\lambda) = \langle (S+\lambda I)(S-\lambda I)^{-1} v, v \rangle .$$

Show that $\operatorname{Re} h$ is a positive harmonic function. [You may *not* invoke the spectral theorem — this is part of a proof of that theorem.]

8: Let Ω be an open convex region in the complex plane. Assume f is a holomorphic function on Ω and the real part of its derivative is positive: $\operatorname{Re}(f'(z)) > 0$ for all $z \in \Omega$.
a) Prove that f is one-to-one.
b) Show by example that the word "convex" cannot be replaced by "connected and simply connected".

9: Let f be a non-constant meromorphic function on the complex plane $\mathbb{C}$ that obeys

$$f(z) = f(z+\sqrt{2}) = f(z+i\sqrt{2}) \ .$$

(In particular, the poles of these three functions coincide.) Assume f has at most one pole in the closed unit disc

$$\overline{D} = \{z : |z| \leq 1\} \ .$$

a) Prove that f has exactly one pole in $\overline{D}$.
b) Prove that this is not a simple pole.

10: Consider the unit sphere in $\mathbb{R}^3$ with poles removed:

$$S := \left\{ \left(\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)\right) : 0 < \phi < \pi \text{ and } \theta \in \mathbb{R} \right\} \ .$$

Give an explicit formula for a conformal map from the complex plane onto S so that horizontal lines are mapped to circles of constant ϕ and vertical lines are mapped to arcs of constant θ.

11: Let f be an analytic function in the open unit disc $D = \{z : |z| < 1\}$ that obeys $|f(z)| \leq 1$ for all $z \in D$. Let g be the restriction of f to the real interval $(0,1)$ and assume

$$\lim_{r \to 1} g(r) = 1$$

and

$$\lim_{r \to 1} g'(r) = 0 \ .$$

Prove that f is constant.

12: Let f be a non-constant meromorphic function in the complex plane. Assume that if f has a pole at the point $z \in \mathbb{C}$, then z is of the form $n\pi$ with an integer $n \in \mathbb{Z}$. Assume that for all non-real z we have the estimate

$$|f(z)| \leq (1 + |\mathrm{Im}(z)|^{-1}) e^{-|\mathrm{Im}(z)|}$$

Prove that for every integer $n \in \mathbb{Z}$, f has a pole at the point πn.