Analysis Qualifying Examination - March 26, 2009

Instructions:

Work any 10 problems and therefore at least 4 from Problems 1 - 6 and at least 4 from Problems 7 - 12. All problems are worth ten points. Full credit on one problem will be better than part credit on two problems.

1. Let f and g be real-valued integrable functions on a measure space $(X, \mathcal{B}, \mu)$, and define

$$F_t = \{x \in X : f(x) > t\}, \ G_t = \{x \in X : g(x) > t\}.$$

Prove

$$\int |f-g| d\mu = \int_{-\infty}^{\infty} \mu\big((F_t \setminus G_t) \cup (G_t \setminus F_t)\big) dt.$$

2. Let H be an infinite dimensional real Hilbert space.

a) Prove the unit sphere $S = \{x \in H : ||x|| = 1\}$ of H is weakly dense in the unit ball $B = \{x \in H : ||x|| \leq 1\}$ of H. (i.e. if $x \in B$, there is a sequence $\{x_n\} \in S$ such that for all $y \in H$, $< x, y >= \lim < x_n, y >$.)

b) Prove there is a sequence T_n of bounded linear operators from H to H such that $||T_n|| = 1$ for all n but $\lim T_n(x) = 0$ for all $x \in H$.

3. Let X be a Banach space and let X^* be its dual Banach space. Prove that if X^* is separable then X is separable.

4. Let $f(x)$ be a non-decreasing function on $[0,1]$. You may assume the theorem that f is differentiable almost everywhere.

a) Prove that $\int_0^1 f'(x)dx \leq f(1) - f(0)$.

Hint: Fatou.

b) Let $\{f_n\}$ be a sequence of non-decreasing functions on the unit interval $[0,1]$, such that the series $F(x) = \sum_{n=1}^{\infty} f_n(x)$ converges for all $x \in [a,b]$. Prove that $F'(x) = \sum_{n=1}^{\infty} f_n'(x)$ almost everywhere on $[0,1]$.

Hint: Let $r_n(x) = \sum_{k \geq n} f_k(x)$. It is enough to show $r_n'(x) \to 0$ a.e. Take a subsequence r_{n_j} such that $r_{n_j}(1) - r_{n_j}(0) \to 0$ and use part (a).

5. Let $I = I_{0,0} = [0,1]$ be the unit interval, and for $n = 0, 1, 2, \ldots$, and $0 \le j \le 2^n - 1$ let

$$I_{n,j} = [j2^{-n}, (j+1)2^{-n}].$$

For $f \in L^1(I, dx)$ define $E_n f(x) = \sum_{j=0}^{2^n-1}\Big(2^n \int_{I_{n,j}} f\,dt\Big)\chi_{I_{n,j}}$.
Prove that if $f \in L^1(I, dx)$ then $\lim_{n\to\infty} E_n f(x) = f(x)$ almost everywhere on I.

6. For $I_{n,j}$ as in Problem 5, define the Haar function $h_{n,j} = 2^{n/2}\Big(\chi_{I_{n+1,2j}} - \chi_{I_{n+1,2j+1}}\Big)$.

a) Carefully draw $I_{2,1}$ and graph $h_{2,1}$.

b) Prove that if $f \in L^2(I)$ with respect to Lebesgue measure and $\int_I f\,dt = 0$, then

$$\int_I |f(x)|^2 dx = \sum_{n,j} \Big| \int f(t) h_{n,j}(t)dt \Big|^2.$$

c) Prove that if $f \in L^1(I)$ and $\int_I f(t)dt = 0$, then almost everywhere on I,

$$f(x) = \sum_{n=1}^{\infty} \sum_{j=0}^{2^n-1} \Big(\int f(t) h_{n,j}(t)dt\Big) h_{n,j}(x).$$

Hint: Compare the n-th partial sum to $E_n f$ from Problem 5.

7. Let μ be a finite positive Borel measure on the complex plane $\mathbb{C}$.

a) Prove that $F(z) = \int_{\mathbb{C}} \frac{1}{z-w} d\mu(w)$ exists for almost all $z \in \mathbb{C}$ and that $\int_K |F(z)|dxdy < \infty$ for every compact $K \subset \mathbb{C}$.

b) Using (a), prove that for almost every horizontal line L (almost everywhere measured by y intercept), and all compact $K \subset L$, $\int_K |F(x+iy)|dx < \infty$.

c) By "almost all squares in $\mathbb{C}$" we mean all squares in $\mathbb{C}$ with sides parallel to the axes except for those squares whose lower left and upper right vertices (z_1, z_2) belong to a Lebesgue measure zero subset of $\mathbb{C}^2$. Prove that for almost all open squares S,

$$\mu(S) = \frac{1}{2\pi i} \int_{\partial S} F(z)dz.$$

Hint: Use b) and the analagous result for vertical lines.

8. Let f be an entire non-constant function that satisfies the functional equation

$$f(1-z) = 1 - f(z)$$

for all $z \in \mathbb{C}$. Show that $f(\mathbb{C}) = \mathbb{C}$.

9. Let $f(z)$ be an analytic function on the entire complex plane $\mathbb{C}$ and assume $f(0) \neq 0$. Let $\{a_n\}$ be the zeros of f, counted with their multiplicities.

a) Let $R > 0$ be such that $|f(z)| > 0$ on $|z| = R$. Prove

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta = \log|f(0)| + \sum_{|a_n|<R} \log(\frac{R}{|a_n|}).$$

b) Assume $|f(z)| \leq Ce^{|z|^\lambda}$ for positive constants C and λ. Prove that

$$\sum (\frac{1}{|a_n|})^{\lambda+\epsilon} < \infty$$

for all $\epsilon > 0$.

Hint: Estimate $\#\{n : |a_n| < R\}$ by using (a) for the circle of radius $2R$.

10. Let $\mathbb{D}$ be the open unit disc and μ be Lebesgue measure on $\mathbb{D}$. Let H be the subspace of $L^2(\mathbb{D}, \mu)$ consisting of holomorphic functions. Show that H is complete.

11. Suppose that $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and injective in some annulus $\{z : r < |z| < 1\}$, where $\mathbb{D}$ is the open unit disc. Show that f is injective in $\mathbb{D}$.

12. Let Q be the closed unit square in the complex plane $\mathbb{C}$ and let R be the closed rectangle in $\mathbb{C}$ with vertices $\{0, 2, i, 2+i\}$. Prove there does *not* exist a surjective homeomorphism $f : Q \to R$ that is conformal on the interior Q^o and that maps corners to corners.