ANALYSIS QUAL: MARCH 24, 2010

Answer at most 10 questions, including at least 4 even numbered questions. On the front of your paper indicate which 10 problem you wish to have graded.

Problem 1. (a) Let $1 \leq p < \infty$. Show that if a sequence of real-valued functions $\{f_n\}_{n\geq 1}$ converges in $L^p(\mathbb{R})$, then it contains a subsequence that converges almost everywhere.

(b) Give an example of a sequence of functions converging to zero in $L^2(\mathbb{R})$ that does not converge almost everywhere.

Problem 2. Let $p_1, p_2, \ldots, p_n$ be distinct points in the complex plane $\mathbb{C}$ and let U be the domain

$$U = \mathbb{C} \setminus \{p_1, \ldots, p_n\}.$$

Let A be the vector space of real harmonic functions on U and let $B \subset A$ be the subspace of real parts of complex analytic functions on U. Find the dimension of the quotient vector space A/B, give a basis for this quotient space, and prove that it is a basis.

Problem 3. For an $f : \mathbb{R} \to \mathbb{R}$ belonging to $L^1(\mathbb{R})$, we define the Hardy-Littlewood maximal function as follows:

$$(Mf)(x) := \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(y)|\, dy.$$

Prove that it has the following property: There is a constant A such that for any $\lambda > 0$,

$$\big|\{x \in \mathbb{R} : (Mf)(x) > \lambda\}\big| \leq \frac{A}{\lambda} \|f\|_{L^1}$$

where $|E|$ denotes the Lebesgue measure of E. If you use a covering lemma, you should prove it.

Problem 4. Let $f(z)$ be a continuous function on the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$ such that $f(z)$ is analytic on the open disk $\{|z| < 1\}$ and $f(0) \neq 0$.

(a) Prove that if $0 < r < 1$ and if $\inf_{|z|=r} |f(z)| > 0$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta \geq \log |f(0)|.$$

(b) Use (a) to prove that $\big|\{\theta \in [0, 2\pi] : f(e^{i\theta}) = 0\}\big| = 0$ where again $|E|$ is the Lebesgue measure of E.

Problem 5. (a) For $f \in L^2(\mathbb{R})$ and a sequence $\{x_n\}_{n\geq 1} \subset \mathbb{R}$ which converges to zero, define

$$f_n(x) := f(x + x_n).$$

Show that $\{f_n\}_{n\geq 1}$ converges to f in L^2 sense.

(b)Let $W \subset \mathbb{R}$ be a Lebesgue measurable set of positive Lebesgue measure. Show that the set of differences

$$W - W = \{x - y : x, y \in W\}$$

contains an open neighborhood of the origin.

Problem 6. Let μ be a finite, positive, regular Borel measure supported on a compact subset of the complex plane $\mathbb{C}$ and define the Newtonian potential of μ to be

$$U_\mu(z) = \int_{\mathbb{C}} \left|\frac{1}{z-w}\right| d\mu(w).$$

(a) Prove that U_μ exists at Lebesgue almost all $z \in \mathbb{C}$ and that

$$\int\int_K U_\mu(z)dxdy < \infty$$

for every compact $K \subset \mathbb{C}$. Hint: Fubini.

(b) Prove that for almost every horizontal or vertical line $L \subset \mathbb{C}$, $\mu(L) = 0$ and $\int_K U_\mu(z)ds < \infty$ for every compact subset $K \subset L$, where ds denotes Lebesgue linear measure on L. Hint: Fubini and (a). (Here a. e. vertical line means the vertical lines through $(x, 0)$ for a.e. $x \in \mathbb{R}$. Likewise for horizontal lines.)

(c) Define the Cauchy potential of μ to be

$$S_\mu(z) = \int_{\mathbb{C}} \frac{1}{z-w} d\mu(w),$$

which trivially exists whenever $U_\mu(z) < \infty$. Let R be a rectangle in $\mathbb{C}$ whose four sides are contained in lines L having the conclusions of (b). Prove that

$$\frac{1}{2\pi i}\int_{\partial R} S_\mu(z)dz = \mu(R).$$

Hint: Fubini and Cauchy.

Problem 7. Let H be a Hilbert space and let E be a closed convex subset of H. Prove that there exists a unique element $x \in E$ such that

$$\|x\| = \inf_{y\in E} \|y\|.$$

Problem 8. Let $F(z)$ be a non-constant meromorphic function on the complex plane $\mathbb{C}$ such that for all $z \in \mathbb{C}$,

$$F(z+1) = F(z) \text{ and } F(z+i) = F(z).$$

Let Q be a square with vertices $z, z+1, z+i$ and $z + (1+i)$ such that F has no zeros and no poles on ∂Q. Prove that inside Q the function F has the same number of zeros as poles (counting multiplicities).

Problem 9. Let
$$A = \{x \in \ell^2 : \sum_{n\geq 1} n|x_n|^2 \leq 1\}.$$

(a) Show that A is compact in the ℓ^2 topology.
(b) Show that the mapping from A to $\mathbb{R}$ defined by
$$x \mapsto \int_0^{2\pi} \Big|\sum_{n\geq 1} x_n e^{in\theta}\Big| \frac{d\theta}{2\pi}$$
achieves its maximum on A.

Problem 10. Let $\Omega \subset \mathbb{C}$ be a connected open set, let $z_0 \in \Omega$, and let $\mathcal{U}$ be the set of postive harmonic functions U on Ω such that $U(z_0) = 1$. Prove for every compact set $K \subset \Omega$ there is a finite constant M (depending on Ω, z_0 and K) such that
$$\sup_{U\in\mathcal{U}} \sup_{z\in K} U(z) \leq M.$$
You may use Harnack's inequality for the disk without proving it, provided you state it correctly.

Problem 11. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support.
(a) Prove there is a constant A such that
$$\|f * \phi\|_{L^q} \leq A\|f\|_{L^p} \quad \text{for all} \quad 1 \leq p \leq q \leq \infty \quad \text{and all} \quad f \in L^p.$$
If you use Young's (convolution) inequality, you should prove it.
(b) Show by example that such a general inequality cannot hold for $p > q$.

Problem 12. Let F be a function from the open unit disk $\mathbb{D} = \{|z| < 1\}$ to $\mathbb{D}$ such that whenever z_1, z_2 and z_3 are distinct points of $\mathbb{D}$ there exists an analytic function f_{z_1,z_2,z_3} from $\mathbb{D}$ into $\mathbb{D}$ such that
$$F(z_j) = f_{z_1,z_2,z_3}(z_j), \; j = 1, 2, 3.$$
Prove that F is analytic at every point of $\mathbb{D}$.

Hint: Fix $z \in \mathbb{D}$ and let $\mathbb{D} \ni z_n \to z, z_n \neq z$. Show that the sequence
$$\frac{F(z_n) - F(z)}{z_n - z}$$
is bounded and then prove that every two of its convergent subsequences have the same limit.

Problem 13. Let X and Y be two Banach spaces. We say that a bounded linear transformation $A : X \to Y$ is *compact* if for every bounded sequence $\{x_n\}_{n\geq 1} \subset X$, the sequence $\{Ax_n\}_{n\geq 1}$ has a convergent subsequence in Y.

Suppose X is reflexive (that is, $(X^*)^* = X$) and X^* is separable. Show that a linear transformation $A : X \to Y$ is compact if and only if for every bounded sequence $\{x_n\}_{n\geq 1} \subset X$, there exists a subsequence $\{x_{n_j}\}$ and a vector $\phi \in X$ such that $x_{n_j} = \phi + r_{n_j}$ and $Ar_{n_j} \to 0$ in Y.