ANALYSIS QUAL: FALL 2012

Answer at most 10 questions. All problems are worth ten points; parts of a problem do not carry equal weight. On the front of your paper indicate which 10 problems you wish to have graded. You must demonstrate adequate knowledge of both real analysis (problems 1–6) and complex analysis (problems 7–12).

Problem 1. Let $1 < p < \infty$ and let $f_n : \mathbb{R}^3 \to \mathbb{R}$ be a sequence of functions such that $\limsup \|f_n\|_{L^p} < \infty$. Show that if f_n converges almost everywhere, then f_n converges weakly in L^p.

Problem 2. Suppose $d\mu$ is a Borel probability measure on the unit circle in the complex plane such that

$$\lim_{n\to\infty} \int_{|z|=1} z^n \, d\mu(z) = 0.$$

For $f \in L^1(d\mu)$ show that

$$\lim_{n\to\infty} \int_{|z|=1} z^n f(z) \, d\mu(z) = 0.$$

Problem 3. Let H be a Hilbert space and let E be a closed convex subset of H. Prove that there exists a unique element $x \in E$ such that

$$\|x\| = \inf_{y\in E} \|y\|.$$

Problem 4. Fix $f \in C(\mathbb{T})$ where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Let s_n denote the n-th partial sum of the Fourier series of f. Prove that

$$\lim_{n\to\infty} \frac{\|s_n\|_{L^\infty(\mathbb{T})}}{\log(n)} = 0.$$

Problem 5. Let $f_n : \mathbb{R}^3 \to \mathbb{R}$ be a sequence of functions such that $\sup_n \|f_n\|_{L^2} < \infty$. Show that if f_n converges almost everywhere to a function $f : \mathbb{R}^3 \to \mathbb{R}$, then

$$\int_{\mathbb{R}^3} \Big| |f_n|^2 - |f_n - f|^2 - |f|^2 \Big| \, dx \to 0.$$

Problem 6. Let $f \in L^1(\mathbb{R})$ and let $\mathcal{M}f$ denote its maximal function, that is,

$$(\mathcal{M}f)(x) = \sup_{0<r<\infty} \frac{1}{2r} \int_{-r}^{r} |f(x-y)|\, dy.$$

By the Hardy–Littlewood maximal function theorem,

$$|\{x \in \mathbb{R} : (\mathcal{M}f)(x) > \lambda\}| \leq 3\lambda^{-1} \|f\|_{L^1} \quad \text{for all } \lambda > 0.$$

Using this show that

$$\limsup_{r \to 0} \frac{1}{2r} \int_{-r}^{r} |f(y) - f(x)|\, dy = 0 \quad \text{for almost every } x \in \mathbb{R}.$$

Problem 7. Let f be function holomorphic in $\mathbb{C}$ and suppose that $f(0) = 0$, $f(1) = 1$, and $f(\mathbb{D}) \subseteq \mathbb{D}$, where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Show that

a) $f'(1) \in \mathbb{R}$,

b) $f'(1) \geq 1$.

Problem 8. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a non-constant holomorphic function such that every zero of f has even multiplicity.

Show that f has a holomorphic square root, i.e., there exists a holomorphic function $g\colon \mathbb{C} \to \mathbb{C}$ such that $f(z) = g(z)^2$ for all $z \in \mathbb{C}$.

Problem 9. Suppose f is a holomorphic function in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $\{x_n\}$ is a sequence of real numbers satisfying $0 < x_{n+1} < x_n < 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = 0$. Show that if $f(x_{2n+1}) = f(x_{2n})$ for all $n \in \mathbb{N}$, then f is a constant function.

Problem 10. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of holomorphic functions on $\mathbb{D}$ satisfying $|f_n(z)| \leq 1$ for all $z \in \mathbb{D}$ and $n \in \mathbb{N}$. Let $A \subseteq \mathbb{D}$ be the set of all $z \in \mathbb{D}$ for which the limit $\lim_{n\to\infty} f_n(z)$ exists. Show that if A has an accumulation point in $\mathbb{D}$, then there exists a holomorphic function f on $\mathbb{D}$ such that $f_n \to f$ locally uniformly on $\mathbb{D}$ as $n \to \infty$.

Problem 11. Find all holomorphic functions $f\colon \mathbb{C} \to \mathbb{C}$ satisfying $f(z+1) = f(z)$ and $f(z+i) = e^{2\pi} f(z)$ for all $z \in \mathbb{C}$.

Problem 12. Let $M \in \mathbb{R}$, $\Omega \subseteq \mathbb{C}$ be a bounded open set, and $u\colon \Omega \to \mathbb{R}$ be a harmonic function.

a) Show that if

$$\limsup_{z \to z_0} u(z) \leq M \tag{1}$$

for all $z_0 \in \partial\Omega$, then $u(z) \leq M$ for all $z \in \Omega$.

b) Show that if u is bounded from above and there exists a finite set $F \subseteq \partial\Omega$ such that (1) is valid for all $z_0 \in \partial\Omega \setminus F$, then the conclusion of (a) is still true, i.e., it follows that $u(z) \leq M$ for all $z \in \Omega$.