ANALYSIS QUALIFYING EXAM, SPRING 2012

Instructions: Work any 10 problems and therefore at least 4 from Problems 1-6 and at least 4 from Problems 7-12. All problems are worth ten points. Full credit on one problem will be better than part credit on two problems. If you attempt more than 10 problems, indicate which 10 are to be graded.

1: Some of the following statements about sequences functions f_n in $L^3([0,1])$ are false. Indicate these and provide an appropriate counterexample.
(a) If f_n converges to f almost everywhere then a subsequence converges to f in L^3.
(b) If f_n converges to f in L^3 then a subsequence converges almost everywhere.
(c) If f_n converges to f in measure (= in probability) then the sequence converges to f in L^3.
(d) If f_n converges to f in L^3 then the sequence converges to f in measure.

2: Let X and Y be topological spaces and $X \times Y$ the Cartesian product endowed with the product topology. $\mathcal{B}(X)$ denotes the Borel sets in X and similarly, $\mathcal{B}(Y)$ and $\mathcal{B}(X \times Y)$.
(a) Suppose $f : X \to Y$ is continuous. Prove that $E \in \mathcal{B}(Y)$ implies $f^{-1}(E) \in \mathcal{B}(X)$.
(b) Suppose $A \in \mathcal{B}(X)$ and $E \in \mathcal{B}(Y)$. Show that $A \times E \in \mathcal{B}(X \times Y)$.

3: Given $f : [0,1] \to \mathbb{R}$ belonging to $L^1(dx)$ and $n \in \{1, 2, 3, \ldots\}$ define

$$f_n(x) = n \int_{k/n}^{(k+1)/n} f(y)\, dy \quad \text{for } x \in [k/n, (k+1)/n) \ \text{ and } \ k = 0, \ldots, n-1.$$

Prove $f_n \to f$ in L^1.

4: Let $S = \{f \in L^1(\mathbb{R}^3) : \int f\, dx = 0\}$.
(a) Show that S is closed in the L^1 topology.
(b) Show that $S \cap L^2(\mathbb{R}^3)$ is a dense subset of $L^2(\mathbb{R}^3)$.

5: State and prove the Riesz Representation Theorem for linear functionals on a (separable) Hilbert space.

6: Suppose $f \in L^2(\mathbb{R})$ and that the Fourier transform obeys $\hat{f}(\xi) > 0$ for almost every ξ. Show that the set of finite linear combinations of translates of f is dense in the Hilbert space $L^2(\mathbb{R})$.

Problems 7-12 on next page.

7: Let $\{u_n(z)\}$ be a sequence of real-valued harmonic functions on $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ that obey

$$u_1(z) \geq u_2(z) \geq u_3(z) \geq \cdots \geq 0 \qquad \text{for all } z \in \mathbb{D}.$$

Prove that $z \mapsto \inf_n u_n(z)$ is a harmonic function on $\mathbb{D}$.

8: Let Ω be the following subset of the complex plane:

$$\Omega := \{x + iy : x > 0,\ y > 0,\ \text{and } xy < 1\}.$$

Give an example of an *unbounded* harmonic function on Ω that extends continuously to $\partial\Omega$ and vanishes there.

9: Prove Jordan's lemma: If $f(z) : \mathbb{C} \to \mathbb{C}$ is meromorphic, $R > 0$, and $k > 0$, then

$$\left| \int_\Gamma f(z) e^{ikz}\, dz \right| \leq \frac{100}{k} \sup_{z \in \Gamma} |f(z)|$$

where Γ is the quarter-circle $z = Re^{i\theta}$ with $0 \leq \theta \leq \pi/2$. (It is possible to replace 100 here by $\pi/2$, but you are not required to prove that.)

10: Let us define the Gamma function via

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt,$$

at least when the integral is absolutely convergent. Show that this function extends to a meromorphic function in the whole complex plane. You cannot use any particular properties of the Gamma function unless you derive them from this definition.

11: Let $P(z)$ be a polynomial. Show that there is an integer n and a second polynomial $Q(z)$ so that

$$P(z)Q(z) = z^n |P(z)|^2 \quad \text{whenever } |z| = 1.$$

12: Show that the only entire function $f(z)$ obeying both

$$|f'(z)| \leq e^{|z|} \quad \text{and} \quad f\Big(\tfrac{n}{\sqrt{1+|n|}}\Big) = 0 \quad \text{for all } n \in \mathbb{Z}$$

is the zero function. Here $'$ denotes differentiation.