Analysis Qualifying Exam

Friday, September 19, 2014, 2:00 p.m.–6:00 p.m.

Students should solve four real analysis problems (numbered 1–6) and four complex analysis problems (numbered 7–12).

Problem 1. Show that

$$A := \{f \in L^3(\mathbb{R}) : \int_{\mathbb{R}} |f(x)|^2 \, dx < \infty\}$$

is a Borel subset of $L^3(\mathbb{R})$.

Problem 2. Construct an $f \in L^1(\mathbb{R})$ so that $f(x+y)$ does *not* converge almost everywhere to $f(x)$ as $y \to 0$. Prove that your f has this property.

Problem 3. Let (f_n) be a bounded sequence in $L^2(\mathbb{R})$ and suppose that $f_n \to 0$ Lebesgue almost everywhere. Show that $f_n \rightharpoonup 0$ in the weak topology on $L^2(\mathbb{R})$.

Problem 4. Given $f \in L^2([0, \pi])$, we say that $f \in \mathcal{G}$ if f admits a representation of the form

$$f(x) = \sum_{n=0}^{\infty} c_n \cos(nx) \quad \text{with} \quad \sum_{n=0}^{\infty} (1+n^2)|c_n|^2 < \infty.$$

Show that if $f \in \mathcal{G}$ and $g \in \mathcal{G}$ then $fg \in \mathcal{G}$.

Problem 5. Let $\phi : [0,1] \to [0,1]$ be continuous and let $d\mu$ be a Borel probability measure on $[0,1]$. Suppose $\mu(\phi^{-1}(E)) = 0$ for every Borel set $E \subseteq [0,1]$ with $\mu(E) = 0$. Show that there is a Borel measurable function $w : [0,1] \to [0,\infty)$ so that

$$\int f \circ \phi(x) \, d\mu(x) = \int f(y) w(y) \, d\mu(y) \quad \text{for all continuous } f : [0,1] \to \mathbb{R}.$$

Problem 6. Let X be a Banach space and X^* its dual space. Suppose X^* is separable (i.e. has a countable dense set); show that X is separable. (You should assume the Axiom of Choice.)

Problem 7. Find an explicit conformal mapping from the upper half-plane slit along the vertical segment,

$$\{z \in \mathbb{C};\ \operatorname{Im} z > 0\} \setminus (0, 0 + ih], \quad h > 0,$$

to the unit disc $\{z \in \mathbb{C};\ |z| < 1\}$.

Problem 8. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Show that

$$|f(z)| \le C e^{a|z|}, \quad z \in \mathbb{C},$$

for some constants C and a if and only if we have

$$\left|f^{(n)}(0)\right| \le M^{n+1}, \quad n = 0, 1, \ldots,$$

for some constant M.

Problem 9. Let $\Omega \subset \mathbb{C}$ be open and connected. Suppose (f_n) is a sequence of injective holomorphic functions defined on Ω, such that $f_n \to f$ locally uniformly in Ω. Show that if f is not constant, then f is also injective in Ω.

Problem 10. Let us introduce a vector space $\mathcal{B}$ defined as follows,

$$\mathcal{B} = \left\{ u : \mathbb{C} \to \mathbb{C}, \quad u \text{ is holomorphic and } \iint_{\mathbb{C}} |u(x+iy)|^2 \, e^{-(x^2+y^2)} \, dx\, dy < \infty \right\}.$$

Show that $\mathcal{B}$ becomes a *complete* space when equipped with the norm

$$\| u \|^2 = \iint_{\mathbb{C}} |u(x+iy)|^2 \, e^{-(x^2+y^2)} \, dx\, dy.$$

Problem 11. Let $\Omega \subset \mathbb{C}$ be open, bounded, and simply connected. Let u be harmonic in Ω and assume that $u \geq 0$. Show the following: for each compact set $K \subset \Omega$, there exists a constant $C_K > 0$ such that

$$\sup_{x \in K} u(x) \leq C_K \inf_{x \in K} u(x). \tag{1}$$

Problem 12. Let $\Omega = \{z \in \mathbb{C}; |z| > 1\}$. Suppose $u : \overline{\Omega} \to \mathbb{R}$ is bounded and continuous on $\overline{\Omega}$ and that it is subharmonic on Ω. Prove the following: If $u(z) \leq 0$ for all $|z| = 1$ then $u(z) \leq 0$ for all $z \in \Omega$.