Analysis Qualifying Examination, March 2014

Instructions: Solve any 10 problems from the list of 12 below. Each problems is worth ten points; parts of a problem do not carry equal weight. You must tell us which 10 problems you want us to grade.

Problem 1: Consider a measure space $(X, \mathcal{X})$ with a sigma-finite measure μ and, for each $t \in \mathbb{R}$, let e_t denote the characteristic function of the interval (t, ∞). Prove that if $f, g\colon X \to \mathbb{R}$ are $\mathcal{X}$-measurable, then $\|f - g\|_{L^1(X)} = \int_{\mathbb{R}} \|e_t \circ f - e_t \circ g\|_{L^1(X)}\,\mathrm{d}t$.

Problem 2: Let $f \in L^1(\mathbb{R}, \mathrm{d}x)$ and $\beta \in (0,1)$. Prove that

$$\int_{\mathbb{R}} \frac{|f(x)|}{|x-a|^{\beta}}\,\mathrm{d}x < \infty$$

for (Lebesgue) a.e. $a \in \mathbb{R}$.

Problem 3: Let $[a,b] \subset \mathbb{R}$ be a finite interval and let $f\colon [a,b] \to \mathbb{R}$ be a bounded Borel measurable function.

(1) Prove that $\{x \in [a,b] : f \text{ continuous at } x\}$ is Borel measurable.
(2) Prove that f is Riemann integrable if and only if it is continuous almost everywhere.

Problem 4: *(a)* Consider a sequence $\{a_n : n \geq 1\} \subset [0,1]$. For $f \in C([0,1])$, let us denote

$$\varphi(f) = \sum_{n=1}^{\infty} 2^{-n} f(a_n).$$

Prove that there is no $g \in L^1([0,1], \mathrm{d}x)$ such that $\varphi(f) = \int f(x)g(x)\,\mathrm{d}x$ is true for all $f \in C([0,1])$.
(b) Each $g \in L^1([0,1], \mathrm{d}x)$ defines a continuous functional T_g on $L^\infty([0,1], \mathrm{d}x)$ by

$$T_g(f) = \int f(x)g(x)\,\mathrm{d}x.$$

Show that there are continuous functionals on $L^\infty([0,1])$ that are not of this form.

Problem 5: Recall that a metric space is separable if it contains a countable dense subset.
(a) Prove that $\ell^1(\mathbb{N})$ and $\ell^2(\mathbb{N})$ are separable Banach spaces but $\ell^\infty(\mathbb{N})$ is not.
(b) Prove that there exists no linear bounded surjective map $T : \ell^2(\mathbb{N}) \to \ell^1(\mathbb{N})$.

Problem 6: Given a Hilbert space $\mathcal{H}$, let $\{a_n\}_{n\geq 1} \subset \mathcal{H}$ be a sequence with $\|a_n\| = 1$ for all $n \geq 1$. Recall that the closed convex hull of $\{a_n\}_{n\geq 1}$ is the closure of the set of all convex combinations of elements in $\{a_n\}_n$.

(a) Show that if $\{a_n\}_n$ spans $\mathcal{H}$ linearly (i.e, any $x \in \mathcal{H}$ is of the form $\Sigma_{k=1}^m c_k a_{n_k}$, for some m and $c_k \in \mathbb{C}$), then $\mathcal{H}$ is finite dimensional.

(b) Show that if $\langle a_n, \xi \rangle \to 0$ for all $\xi \in \mathcal{H}$, then 0 is in the closed convex hull of $\{a_n\}_n$.

Problem 7: Characterize all entire functions f with $|f(z)| > 0$ for $|z|$ large and

$$\limsup_{z\to\infty} \frac{|\log |f(z)||}{|z|} < \infty$$

Problem 8: Construct a non-constant entire function $f(z)$ such that the zeros of f are simple and coincide with the set of all (positive) natural numbers.

Problem 9: Prove Hurwitz' Theorem: Let $\Omega \subset \mathbb{C}$ be a connected open set and $f_n, f : \Omega \to \mathbb{C}$ holomorphic functions. Assume that $f_n(z)$ converges uniformly to $f(z)$ on compact subsets of Ω. Prove that if $f_n(z) \neq 0$, $\forall z \in \Omega$, $\forall n$, then either f is identically equal to 0, or $f(z) \neq 0$, $\forall z \in \Omega$.

Problem 10: Let $\alpha \in [0,1] \setminus \mathbb{Q}$ and let $\{a_n\} \in \ell^1(\mathbb{N})$ with $a_n \neq 0$ for all $n \geq 1$. Set $\mathbb{D} := \{z \in \mathbb{C} \colon |z| < 1\}$. Show that

$$f(z) = \sum_{n\geq 1} \frac{a_n}{z - e^{i\alpha n}}, \qquad z \in \mathbb{D},$$

converges and defines a function that is analytic in $\mathbb{D}$ which does not admit an analytic continuation to any domain larger than $\mathbb{D}$.

Problem 11: For each $p \in (-1,1)$, compute the improper Riemann integral

$$\int_0^\infty \frac{x^p}{x^2+1}\,dx$$

Problem 12: Compute the number of zeros, including multiplicity, of $f(z) = z^6 + iz^4 + 1$ in the upper half plane in $\mathbb{C}$.