Analysis Qualifying Exam, September 25, 2015, 2:00 p.m. — 6:00 p.m.

Students should solve four real analysis problems (numbered 1–6) and four complex analysis problems (numbered 7–12).

Problem 1. Let g_n be a sequence of measurable functions on $\mathbf{R}^d$, such that $|g_n(x)| \le 1$ for all x, and assume that $g_n \to 0$ almost everywhere. Let $f \in L^1(\mathbf{R}^d)$. Show that the sequence

$$f * g_n(x) = \int f(x-y) g_n(y)\, dy \to 0$$

uniformly on each compact subset of $\mathbf{R}^d$, as $n \to \infty$.

Problem 2. Let $f \in L^p(\mathbf{R})$, $1 < p < \infty$, and let $a \in \mathbf{R}$ be such that $a > 1 - \frac{1}{p}$. Show that the series

$$\sum_{n=1}^{\infty} \int_n^{n+n^{-a}} |f(x+y)|\, dy$$

converges for almost all $x \in \mathbf{R}$.

Problem 3. Let $f \in L^1_{\text{loc}}(\mathbf{R}^d)$ be such that for some $0 < p < 1$, we have

$$\left| \int f(x) g(x)\, dx \right| \le \left(\int |g(x)|^p\, dx \right)^{\frac{1}{p}},$$

for all $g \in C_0(\mathbf{R}^d)$. Show that $f(x) = 0$ a.e. Here $C_0(\mathbf{R}^d)$ is the space of continuous functions with compact support on $\mathbf{R}^d$.

Problem 4. Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space and assume that (e_n) is an orthonormal system in $\mathcal{H}$. Let (f_n) be another orthonormal system which is complete, i.e. the closure of the span of the (f_n) is all of $\mathcal{H}$.

- Show that if $\sum_{n=1}^{\infty} || f_n - e_n ||^2 < 1$ then the orthonormal system (e_n) is also complete.
- Assume that we only have $\sum_{n=1}^{\infty} || f_n - e_n ||^2 < \infty$. Prove that it is still true that (e_n) is complete.

Problem 5. A function $f \in C([0,1])$ is called Hölder continuous of order $\delta > 0$ if there is a constant $C > 0$ such that $|f(x) - f(y)| \le C\, |x-y|^{\delta}$, $x, y \in [0,1]$. Show that the Hölder continuous functions form a set of the first category (a meager set) in $C([0,1])$.

Problem 6. Let $u \in L^2(\mathbf{R}^d)$ and let us say that $u \in H^{1/2}(\mathbf{R}^d)$ (a Sobolev space) if

$$\left(1 + |\xi|^{1/2}\right) \widehat{u}(\xi) \in L^2(\mathbf{R}^d).$$

Here $\widehat{u}$ is the Fourier transform of u. Show that $u \in H^{1/2}(\mathbf{R}^d)$ if and only if

$$\iint \frac{|u(x+y)-u(x)|^2}{|y|^{d+1}}\,dx\,dy < \infty.$$

Problem 7. Assume that $f(z)$ is analytic in $\{z : |z| < 1\}$ and continuous on $\{z : |z| \leq 1\}$. If $f(z) = f(1/z)$ when $|z| = 1$, prove that $f(z)$ is constant.

Problem 8. Assume that $f(z)$ is an entire function that is 2π–periodic in the sense that $f(z+2\pi) = f(z)$, and

$$|f(x+iy)| \leq Ce^{\alpha|y|},$$

for some $C > 0$, where $0 < \alpha < 1$. Prove that f is constant.

Problem 9. Let (f_j) be a sequence of entire functions such that, writing $z = x + iy$, we have

$$\iint_{\mathbf{C}} |f_j(z)|^2\, e^{-|z|^2}\,dx\,dy \leq C, \quad j = 1, 2, \ldots,$$

for some constant $C > 0$. Show that there exists a subsequence (f_{j_k}) and an entire function f such that we have

$$\iint_{\mathbf{C}} |f_{j_k}(z) - f(z)|^2\, e^{-2|z|^2}\,dx\,dy \to 0, \quad k \to \infty.$$

Problem 10. Use the Residue Theorem to prove that

$$\int_0^\infty e^{\cos x}\sin(\sin x)\frac{dx}{x} = \frac{\pi}{2}(e-1).$$

Use a large semicircle as part of the contour.

Problem 11. Let $\Omega = \{(x,y) \in \mathbf{R}^2; x > 0 y > 0\}$ and let u be subharmonic in Ω, continuous in $\overline{\Omega}$, such that

$$u(x,y) \leq |x+iy|,$$

for large $(x,y) \in \Omega$. Assume that

$$u(x,0) \leq ax, \quad u(0,y) \leq by, \quad x,y \geq 0,$$

for some $a, b > 0$. Show that

$$u(x,y) \leq ax + by, \quad (x,y) \in \Omega.$$

Problem 12. Find a function $u(x,y)$ harmonic in the region between the circles $|z| = 2$ and $|z-1| = 1$ which equals 1 on the outer circle and 0 on the inner circle (except at the point where the circles are tangent to each other).