Analysis Qualifying Exam, March 25, 2015, 9:00 a.m. — 1:00 p.m.

Students should solve four real analysis problems (numbered 1–6) and four complex analysis problems (numbered 7–12).

Problem 1. Let $f \in L^1(\mathbf{R})$. Show that

$$\lim_{n\to\infty} \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} f(x)\,dx \right| = \int |f(x)|\,dx.$$

Problem 2. Let $f \in L^2_{\text{loc}}(\mathbf{R}^n)$, $g \in L^3_{\text{loc}}(\mathbf{R}^n)$. Assume that for all real $r \geq 1$, we have

$$\int_{r\leq|x|\leq 2r} |f(x)|^2\,dx \leq r^a, \quad \int_{r\leq|x|\leq 2r} |g(x)|^3\,dx \leq r^b.$$

Here $a, b \in \mathbf{R}$ are such that $3a + 2b + n < 0$. Show that $fg \in L^1(\mathbf{R}^n)$.

Problem 3. Let $f \in L^1_{\text{loc}}(\mathbf{R}^n)$ and let

$$Mf(x) = \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y)|\,dy$$

be the Hardy-Littlewood maximal function.

- Show that

$$m(\{x : Mf(x) > s\}) \leq \frac{C_n}{s} \int_{|f(x)|>s/2} |f(x)|\,dx, \quad s > 0,$$

  where the constant C_n depends on n only. The Hardy-Littlewood maximal theorem may be used.

- Prove that if $\varphi \in C^1(\mathbf{R})$, $\varphi(0) = 0$, and $\varphi' > 0$ then

$$\int \varphi(Mf(x))\,dx \leq C_n \int |f(x)| \left(\int_{0<t<2|f(x)|} \frac{\varphi'(t)}{t}\,dt \right) dx.$$

Problem 4. Let $f \in L^1_{\text{loc}}(\mathbf{R})$ be 2π-periodic. Show that linear combinations of the translates $f(x-a)$, $a \in \mathbf{R}$, are dense in $L^1((0, 2\pi))$ if and only if each Fourier coefficient of f is $\neq 0$.

Problem 5. Let $u \in L^2(\mathbf{R})$ and let us set

$$U(x,\xi) = \int e^{-(x+i\xi-y)^2/2} u(y)\,dy, \quad x, \xi \in \mathbf{R}.$$

Show that $U(x,\xi)$ is well defined on $\mathbf{R}^2$ and that there exists a constant $C > 0$ such that for all $u \in L^2(\mathbf{R})$, we have

$$\iint |U(x,\xi)|^2 \, e^{-\xi^2} \, dx \, d\xi = C \int |u(y)|^2 \, dy.$$

Determine C explicitly.

Problem 6. When B_1 and B_2 are Banach spaces, we say that a linear operator $T : B_1 \to B_2$ is compact if for any bounded sequence (x_n) in B_1, the sequence (Tx_n) has a convergent subsequence. Show that if T is compact then $\operatorname{Im} T$ has a dense countable subset.

Problem 7. Let $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ and $\mathbf{C}^+ = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$. Suppose $f_n : \mathbf{D} \to \mathbf{C}^+$ is a sequence of holomorphic functions and $f_n(0) \to 0$ as $n \to \infty$. Show that $f_n(z) \to 0$ uniformly on compact subsets of $\mathbf{D}$.

Problem 8. Let $f : \mathbf{C} \to \mathbf{C}$ be holomorphic and suppose

$$\sup_{x \in \mathbf{R}} \left\{ |f(x)|^2 + |f(ix)|^2 \right\} < \infty \quad \text{and} \quad |f(z)| \le e^{|z|} \quad \text{for all } z \in \mathbf{C}.$$

Deduce that $f(z)$ is constant.

Problem 9. Let $\Omega = \{z \in \mathbf{C} : |z| > 1 \text{ and } \operatorname{Re} z > -2\}$. Suppose $u : \overline{\Omega} \to \mathbf{R}$ is bounded, continuous, and harmonic on Ω and also that $u(z) = 1$ when $|z| = 1$ and that $u(z) = 0$ when $\operatorname{Re}(z) = -2$. Determine $u(2)$.

Problem 10. Determine

$$\int_{-\infty}^{\infty} \frac{dy}{(1 + y^2)(1 + [x - y]^2)}$$

for all $x \in \mathbf{R}$. Justify all manipulations.

Problem 11. Let $\Omega = \{z \in \mathbf{C} : 0 < |z| < 1\}$. Prove that for every bounded harmonic function $u : \Omega \to \mathbf{R}$ there is a harmonic function $v : \Omega \to \mathbf{R}$ obeying

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Problem 12. Find all entire functions $f : \mathbf{C} \to \mathbf{C}$ that obey

$$f'(z)^2 + f(z)^2 = 1.$$

Prove that your list is exhaustive.