Analysis Qualifying Exam, March 24, 2016, 9:00 a.m. — 1:00 p.m.

Students should solve four real analysis problems (numbered 1–6) and four complex analysis problems (numbered 7–12).

Problem 1. Let

$$K_t(x) = (4\pi t)^{-3/2} e^{-|x|^2/4t}, \quad x \in \mathbf{R}^3,\ t > 0,$$

where $|x|$ is the Euclidean norm of $x \in \mathbf{R}^3$.

- Show that the linear map

  $$L^3(\mathbf{R}^3) \ni f \mapsto t^{1/2} K_t * f \in L^\infty(\mathbf{R}^3)$$

  is bounded, uniformly in $t > 0$. Here

  $$K_t * f(x) = \int_{\mathbf{R}^3} K_t(x-y) f(y)\, dy$$

  is the convolution.

- Prove that $t^{1/2} \| K_t * f \|_{L^\infty} \to 0$ as $t \to 0$, for $f \in L^3(\mathbf{R}^3)$.

Problem 2. Let $f \in L^1(\mathbf{R})$. Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} f(x - \sqrt{n})$$

converges absolutely for almost all $x \in \mathbf{R}$.

Problem 3. Let $f \in L^1_{\text{loc}}(\mathbf{R})$ be real valued and assume that for each integer $n > 0$, we have

$$f\left(x + \frac{1}{n}\right) \geq f(x),$$

for almost all $x \in \mathbf{R}$. Show that for each real number $a \geq 0$ we have

$$f(x+a) \geq f(x),$$

for almost all $x \in \mathbf{R}$.

Problem 4. Let V_1 be a finite-dimensional subspace of the Banach space V. Show that there exists a continuous projection $P : V \to V_1$, i.e., a continuous linear map $P : V \to V$ such that $P^2 = P$ and the range of P is equal to V_1.

Problem 5. For $f \in C_0^\infty(\mathbf{R}^2)$ define $u(x,t)$ by

$$u(x,t) = \int_{\mathbf{R}^2} e^{ix\cdot\xi} \frac{\sin(t\,|\xi|)}{|\xi|} f(\xi)\,d\xi, \quad x \in \mathbf{R}^2, \quad t > 0.$$

Show that $\lim_{t\to\infty} ||\,u(\cdot,t)\,||_{L^2} = \infty$ for a set of f that is dense in $L^2(\mathbf{R}^2)$.

Problem 6. Suppose that $\{\phi_n\}$ is an orthonormal system of continuous functions in $L^2([0,1])$ and let S be the closure of the span of $\{\phi_n\}$. If $\sup_{f\in S\setminus\{0\}} \frac{||\,f\,||_\infty}{||\,f\,||_2}$ is finite, prove that S is finite dimensional.

Problem 7. Determine

$$\int_0^\infty \frac{x^{a-1}}{x+z}\,dx,$$

for $0 < a < 1$ and $\operatorname{Re} z > 0$. Justify all manipulations.

Problem 8. Let $\mathbf{C}_+ = \{z \in \mathbf{C};\, \operatorname{Im} z > 0\}$ and let $f_n : \mathbf{C}_+ \to \mathbf{C}_+$ be a sequence of holomorphic functions. Show that unless $|f_n| \to \infty$ uniformly on compact subsets of $\mathbf{C}_+$, there exists a subsequence converging uniformly on compact subsets of $\mathbf{C}_+$.

Problem 9. Let $f : \mathbf{C} \to \mathbf{C}$ be entire and assume that $|f(z)| = 1$ when $|z| = 1$. Show that $f(z) = Cz^m$, for some integer $m \geq 0$ and $C \in \mathbf{C}$ with $|C| = 1$.

Problem 10. Does there exist a function $f(z)$ holomorphic in the disk $|z| < 1$ such that $\lim_{|z|\to 1} |f(z)| = \infty$? Either find one or prove that none exist.

Problem 11. Assume that $f(z)$ is holomorphic on $|z| < 2$. Show that

$$\max_{|z|=1} \left| f(z) - \frac{1}{z} \right| \geq 1.$$

Problem 12. [1]

- (a) Find a real-valued harmonic function v defined on the disk $|z| < 1$ such that $v(z) > 0$ and $\lim_{z\to 1} v(z) = \infty$.
- (b) Let u be a real-valued harmonic function in the disk $|z| < 1$ such that $u(z) \leq M < \infty$ and $\limsup_{r\to 1} u(re^{i\theta}) \leq 0$ for all $\theta \in (0, 2\pi)$. Show that $u(z) \leq 0$. The function in part (a) is useful here.

[1] The following version of the problem is better than the original: Let u be a real-valued harmonic function in the disk $|z| < 1$ such that $u(z) \leq M < \infty$ and $\lim_{r\to 1} u(re^{i\theta}) \leq 0$ for almost all θ. Show that $u(z) \leq 0$.