

## ANALYSIS QUAL: SEPTEMBER 2018

Answer at most 10 questions, including at least 4 from questions 1 - 6 and at least 4 from questions 7 - 12. On the front of your paper indicate which 10 problem you wish to have graded.

**Problem 1.** Let  $\{f_n\}$  be a sequence of real-valued Lebesgue measurable functions on  $\mathbb{R}$ , and let  $f$  be another such function. Assume that

- (i)  $f_n \rightarrow f$ , Lebesgue almost everywhere,
- (ii)  $\int_{\mathbb{R}} |x| |f_n(x)| dx \leq 100$ , for all  $n$ , and
- (iii)  $\int_{\mathbb{R}} |f_n(x)|^2 dx \leq 100$ , for all  $n$ .

Prove that  $f_n \in L^1$  for all  $n$ , that  $f \in L^1$ , and that  $\|f_n - f\|_{L^1} \rightarrow 0$ . Also show that neither assumption (ii) nor assumption (iii) can be omitted while making these deductions.

**Problem 2.** Let  $(X, \rho)$  be a compact metric space which has at least two points, and let  $C(X)$  be the space of continuous functions  $X \rightarrow \mathbb{R}$  with the uniform norm  $\|f\| = \sup_X |f(x)|$ . Let  $D$  be a dense subset of  $X$  and for each  $y \in D$  define  $f_y \in C(X)$  by

$$f_y(x) = \rho(x, y).$$

Let  $A$  be the subalgebra of  $C(X)$  generated by the collection  $\{f_y : y \in D\}$  (with pointwise addition and multiplication of functions).

- (a) Prove that  $A$  is dense in  $C(X)$  under the uniform norm.
- (b) Prove that  $C(X)$  is separable.

**Problem 3.** Let  $(X, \rho)$  be a compact metric space and let  $P(X)$  be the set of all probability measures on the Borel sigma-algebra of  $X$  (i.e.  $\mu \in P(X)$  if  $\mu$  is a positive Borel measure and  $\mu(X) = 1$ ). Assume  $\{\mu_n\}$  is a sequence in  $P(X)$  and  $\mu$  is another element of  $P(X)$  such that for all continuous  $f : X \rightarrow \mathbb{R}$

$$\int_X f(x) d\mu_n \rightarrow \int_X f(x) d\mu \quad (n \rightarrow \infty).$$

Prove that

$$\mu_n(E) \rightarrow \mu(E) \quad (n \rightarrow \infty)$$

whenever  $E$  is a Borel subset of  $X$  such that  $\mu(\overline{E}) = \mu(E^o)$ , where  $\overline{E}$  is the closure of  $E$  and  $E^o$  is the interior of  $E$ .

**Problem 4.** Let  $\mathbb{T}$  be the unit circle in the complex plane  $\mathbb{C}$  and for each  $\alpha \in \mathbb{T}$  define the rotation map  $R_\alpha : \mathbb{T} \rightarrow \mathbb{T}$  by  $R_\alpha(z) = \alpha z$ . A Borel probability measure  $\mu$  on  $\mathbb{T}$  is called  $\alpha$ -**invariant** if  $\mu(R_\alpha(E)) = \mu(E)$  for all Borel sets  $E \subset \mathbb{T}$ .

(a) Let  $m$  be Lebesgue measure on  $\mathbb{T}$  (defined, for instance, by indentifying  $\mathbb{T}$  with  $[0, 1)$  through the exponential function). Show that for every  $\alpha \in \mathbb{T}$ ,  $m$  is  $\alpha$ -invariant.

(b) Prove that if  $\alpha$  is not a root of unity, then the set of powers  $\{\alpha^n : n \in \mathbb{Z}\}$  is dense in  $\mathbb{T}$ .

(c) Prove that, if we fix a single  $\alpha \in \mathbb{T}$  which is not a root of unity, then  $m$  is the only  $\alpha$ -invariant Borel probability measure on  $\mathbb{T}$ .

**Problem 5.** Let  $\{f_n\}$  be a sequence of continuous real-valued functions on  $[0, 1]$  and suppose  $f_n(x)$  converges to another real valued function  $f(x)$  at **every**  $x \in [0, 1]$ .

(a) Prove that for every  $\varepsilon > 0$  there is a dense subset  $D_\varepsilon \subset [0, 1]$  such that if  $x \in D_\varepsilon$  then there are an open interval  $I \ni x$  and a positive integer  $N_x$  such that for all  $n > N_x$

$$\sup_{y \in I} |f_n(y) - f(y)| \leq \varepsilon.$$

Hint: Consider the closed sets

$$F_{N,\varepsilon} = \{y \in [0, 1] : |f_n(y) - f_m(y)| \leq \varepsilon, \forall n, m > N\}.$$

(b) Prove that  $f$  cannot be the characteristic function  $\chi_{\mathbb{Q} \cap [0,1]}$  where  $\mathbb{Q}$  is the rational numbers.

**Problem 6.** Let  $f \in L^2(\mathbb{R})$  and assume the Fourier transform satisfies  $|\hat{f}(\xi)| > 0$  for Lebesgue almost every  $\xi \in \mathbb{R}$ . Prove the set of finite linear combinations of the translates  $f_y(x) = f(x - y)$  is norm dense in  $L^2(\mathbb{R})$ .

**Problem 7.** Let  $f(z)$  be an analytic function on the entire complex plane  $\mathbb{C}$  such that the function  $U(z) = \log |f(z)|$  is Lebesgue area integrable:  $\int_{\mathbb{C}} |U(z)| dx dy < \infty$ . Prove  $f$  is constant.

**Problem 8.** Let  $\mathcal{D}$  be the space of analytic functions  $f(z)$  on the unit disc  $\mathbb{D} = \{z : |z| < 1\}$  such that  $f(0) = 0$  and  $\int_{\mathbb{D}} |f'(z)|^2 dx dy < \infty$ .

(a) Prove  $\mathcal{D}$  is complete in the norm

$$\|f\| = \left( \int_{\mathbb{D}} |f'(z)|^2 dx dy \right)^{1/2}.$$

(b) Give a necessary and sufficient condition on the coefficients  $a_n$  for the function  $f(z) = \sum_{n \geq 1} a_n z^n$  to belong to  $\mathcal{D}$ .

**Problem 9.** Consider the meromorphic function  $g(z) = -\pi z \cot(\pi z)$  on the entire plane  $\mathbb{C}$ .

(a) Find all poles of  $g$  and determine the residue of  $g$  at each pole.

(b) In the Taylor series representation  $\sum_{k=0}^{\infty} a_k z^k$  of  $g(z)$  about  $z = 0$ , show that for each  $k \geq 1$

$$a_{2k} = \sum_{n \geq 1} \frac{2}{n^{2k}}.$$

**Problem 10.** For  $-1 < \beta < 1$  evaluate

$$\int_0^{\infty} \frac{x^{\beta}}{1+x^2} dx.$$

**Problem 11.** An **analytic Jordan curve** is a set of the form

$$\Gamma = f(\{|z| = 1\})$$

where  $f$  is analytic and  $1 - 1$  on an annulus  $\{r < |z| < 1/r\}$ ,  $0 < r < 1$ .

Let  $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$  be the Riemann sphere, let  $N < \infty$ , and let  $\Omega \subset \mathbb{C}^*$  be a domain for which  $\partial\Omega$  has  $N$  connected components, none of which are single points. Prove there is a conformal (i.e. one-to-one analytic) mapping from  $\Omega$  onto a domain bounded by  $N$  pairwise disjoint analytic Jordan curves.

**Problem 12.** If  $\alpha \in \mathbb{C}$  satisfies  $0 < |\alpha| < 1$  and if  $n \in \mathbb{N} = \{1, 2, 3, \dots\}$  show that the equation

$$e^z (z - 1)^n = \alpha$$

has exactly  $n$  simple roots in the half plane  $\{z : \operatorname{Re} z > 0\}$ .