ANALYSIS QUAL: MARCH 28, 2018

Please be reminded that to pass the exam you need to show mastery of both real and complex analysis. Please choose at most 10 questions to answer, including at least 4 from problems 1–6 and 4 from problems 7–12. On the front of your paper indicate which 10 problems you wish to have graded.

Problem 1. Suppose $f \in L^1(\mathbb{R})$ satisfies

$$\limsup_{h\to 0} \int_{\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} \right| dx = 0.$$

Show that $f = 0$ almost everywhere.

Problem 2. Given $f \in L^2(\mathbb{R})$ and $h > 0$ we define

$$Q(f,h) = \int_{\mathbb{R}} \frac{2f(x) - f(x+h) - f(x-h)}{h^2} f(x)\, dx.$$

(a) Show that

$$Q(f,h) \geq 0 \quad \text{for all } f \in L^2(\mathbb{R}) \text{ and all } h > 0.$$

(b) Show that the set

$$E = \big\{ f \in L^2(\mathbb{R}) : \limsup_{h\to 0} Q(f,h) \leq 1 \big\}$$

is closed in $L^2(\mathbb{R})$.

Problem 3. Suppose $f \in L^1(\mathbb{R})$ satisfies

$$\limsup_{\varepsilon\to 0} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\big|f(x)f(y)\big|}{|x-y|^2 + \varepsilon^2}\, dx\, dy < \infty.$$

Show that $f = 0$ almost everywhere.

Problem 4. (a) Fix $1 < p < \infty$. Show that

$$f \mapsto [Mf](x,y) = \sup_{r>0,\rho>0} \frac{1}{4r\rho} \int_{-r}^{r} \int_{-\rho}^{\rho} f(x+h, y+\ell)\, dh\, d\ell$$

is bounded on $L^p(\mathbb{R}^2)$.

(b) Show that

$$\big[A_r f\big](x,y) = \frac{1}{4r^3} \int_{-r}^{r} \int_{-r^2}^{r^2} f(x+h, y+\ell)\, dh\, d\ell$$

converges to f a.e. in the plane as $r \to 0$.

Problem 5. Let μ be a real-valued Borel measure on $[0,1]$ such that

$$\int_0^1 \frac{1}{x+t}\, d\mu(t) = 0 \quad \text{for all } x > 1.$$

Show that $\mu = 0$.

Problem 6. Let $\mathbb{T}$ denote the unit circle in the complex plane and let $\mathcal{P}(\mathbb{T})$ denote the space of Borel probability measures on $\mathbb{T}$ and $\mathcal{P}(\mathbb{T}\times\mathbb{T})$ denote the space of Borel probability measures on $\mathbb{T}\times\mathbb{T}$. Fix $\mu,\nu \in \mathcal{P}(\mathbb{T})$ and define

$$\mathcal{M} = \Big\{ \gamma \in \mathcal{P}(\mathbb{T}\times\mathbb{T}) : \iint_{\mathbb{T}\times\mathbb{T}} f(x)g(y)\, d\gamma(x,y) = \int_{\mathbb{T}} f(x)\, d\mu(x) \cdot \int_{\mathbb{T}} g(y)\, d\nu(y) \\ \text{for all } f,g \in C(\mathbb{T}) \Big\}.$$

Show that $F : \mathcal{M} \to \mathbb{R}$ defined by

$$F(\gamma) = \iint_{\mathbb{T}\times\mathbb{T}} \sin^2\!\big(\tfrac{\theta-\phi}{2}\big)\, d\gamma\big(e^{i\theta}, e^{i\phi}\big)$$

achieves its minimum on $\mathcal{M}$.

Problem 7. Let $F : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ be (jointly) continuous and holomorphic in each variable separately. Show that $z \mapsto F(z,z)$ is holomorphic.

Problem 8. Determine the supremum of

$$\big|\tfrac{\partial u}{\partial x}(0,0)\big|$$

among all harmonic functions $u : \mathbb{D} \to [0,1]$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Prove that your answer is correct.

Problem 9. Consider the formal product

$$\prod_{n=1}^{\infty} \big(1 + \tfrac{1}{n}\big)^z \big(1 - \tfrac{z}{n}\big)$$

(a) Show that the product converges for any $z \in (-\infty, 0)$.
(b) Show that the resulting function extends from this interval to an entire function of $z \in \mathbb{C}$.

Problem 10. Let $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and let $\Omega = \mathbb{C}^* \setminus \{0,1\}$. Let $f : \Omega \to \Omega$ be a holomorphic function.

(a) Prove that if f is injective then $f(\Omega) = \Omega$.

(b) Make a list of all such injective functions f.

Problem 11. For $R > 1$ let A_R be the annulus $\{1 < |z| < R\}$. Assume there is a conformal (i.e. injective, holomorphic) mapping F from A_{R_1} onto A_{R_2}. Prove that $R_1 = R_2$.

Problem 12. Let $f(z)$ be bounded and holomorphic on the unit disc $\mathbb{D} = \{|z| < 1\}$. Prove that for any $w \in \mathbb{D}$ we have

$$f(w) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{f(z)}{(1 - \overline{z}w)^2} \, dA(z),$$

where $dA(z)$ means integration with respect to Lebesgue measure.