Analysis Qualifying Examination, Fall 2019

Instructions: Solve any 10 problems from the list of 12 below. Each problems is worth ten points; parts of a problem do not carry equal weight. You must tell us which 10 problems you want us to grade. The use of the Axiom of Choice is permitted.

Problem 1: Given σ-finite measures $\mu_1, \mu_2, \nu_1, \nu_2$ on a measurable space $(X, \mathcal{X})$, suppose that $\mu_1 \ll \nu_1$ and $\mu_2 \ll \nu_2$. Prove that the product measures $\mu_1 \otimes \mu_2$ and $\nu_1 \otimes \nu_2$ on $(X \times X, \mathcal{X} \otimes \mathcal{X})$ satisfy $\mu_1 \otimes \mu_2 \ll \nu_1 \otimes \nu_2$ and the Random-Nikodym derivatives obey

$$\frac{\mathrm{d}(\mu_1 \otimes \mu_2)}{\mathrm{d}(\nu_1 \otimes \nu_2)}(x,y) = \frac{\mathrm{d}\mu_1}{\mathrm{d}\nu_1}(x)\frac{\mathrm{d}\mu_2}{\mathrm{d}\nu_2}(y)$$

for $\nu_1 \otimes \nu_2$ almost every $(x,y) \in X \times X$.

Problem 2: Let μ be a finite Borel measure on $\mathbb{R}$ with $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$ and let $\varphi(t) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}\mu(x)$. Prove that

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\varphi(t)|^2\,\mathrm{d}t = 0$$

Problem 3: Consider a measure space $(X, \mathcal{X})$ with σ-finite measure μ and let $p \in (1,\infty)$. Let $L^{p,\infty}$ be the set of measurable $f\colon X \to \mathbb{R}$ with $[f]_p = \sup_{t>0} t\mu(|f| > t)^{1/p}$ finite. Let

$$\|f\|_{p,\infty} = \sup_{\substack{E\in\mathcal{X}\\ \mu(E)\in(0,\infty)}} \frac{1}{\mu(E)^{1-1/p}}\int_E |f|\,\mathrm{d}\mu$$

Prove that there exist $c_1, c_2 \in (0,\infty)$ — which may depend on p and μ — such that

$$\forall f \in L^{p,\infty}\colon \quad c_1[f]_p \le \|f\|_{p,\infty} \le c_2[f]_p$$

Problem 4: Let $A \subseteq \mathbb{R}$ be measurable with positive Lebesgue measure. Prove that the set $A - A = \{z - y : z, y \in A\}$ has non-empty interior. *Hint*: Consider the function $\varphi(x) = \int \chi_A(x+y)\chi_A(y)\,\mathrm{d}y$, where χ_A is the characteristic function of A.

Problem 5: Prove the following claim: Let $\mathcal{H}$ be a Hilbert space with the scalar product of x and y denoted by (x,y) and let $A, B\colon \mathcal{H} \to \mathcal{H}$ be (everywhere-defined) linear operators with

$$\forall x, y \in \mathcal{H}\colon \quad (Bx, y) = (x, Ay)$$

Then A and B are both bounded (and thus continuous).

Problem 6: Recall that $\ell^\infty(\mathbb{N}) = \{x = \{x_n\}_{n\geq 1} \colon \sup_{n\geq 1} |x_n| < \infty\}$ is a Banach space (over $\mathbb{R}$) with respect to the norm $\|x\|_\infty = \sup_{n\geq 1} |x_n|$.

(1) Prove that there exists a continuous linear functional ϕ on $\ell^\infty(\mathbb{N})$ such that
$$\phi(x) = \lim_{n\to\infty} x_n$$
whenever the limit exists.

(2) Prove that this ϕ is not unique.

Problem 7: Let $J \subseteq \mathbb{R}$ be a compact interval, and let μ be a finite Borel measure whose support lies in J. For $z \in \mathbb{C} \setminus J$ define
$$F_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\, \mathrm{d}\mu(t)$$
Prove that the mapping $\mu \mapsto F_\mu$ is one-to-one.

Problem 8: A function $f\colon \mathbb{C} \to \mathbb{C}$ is entire and has the property that $|f(z)| = 1$ when $|z| = 1$. Prove that $f(z) = az^n$ for some integer $n \geq 0$ and some $a \in \mathbb{C}$ with $|a| = 1$.

Problem 9: Determine the number of zeros of the polynomial
$$P(z) = z^6 - 6z^2 + 10z + 2$$
in the annulus $\{z \in \mathbb{C} \colon 1 < |z| < 2\}$. Prove your claim.

Problem 10: Evaluate
$$\lim_{x\to\infty} \int_0^x \sin(t^2)\, \mathrm{d}t$$
Justify all steps.

Problem 11: Find a conformal map of the domain
$$D = \left\{z \in \mathbb{C} \colon |z-1| < \sqrt{2},\ |z+1| < \sqrt{2}\right\}$$
onto the open unit disc centered at the origin. It suffices to write this map as a composition of explicit conformal maps.

Problem 12: Show that
$$F(z) = \int_1^\infty \frac{t^z}{\sqrt{1+t^3}}\, \mathrm{d}t$$
is well defined (by the integral) and analytic in $\{z \in \mathbb{C} \colon \mathrm{Re} z < \frac{1}{2}\}$, and admits a meromorphic continuation to the region $\{z \in \mathbb{C} \colon \mathrm{Re} z < \frac{3}{2}\}$.