ANALYSIS QUALIFYING EXAM, March 28, 2019

Answer at most 10 questions, including at least 4 from questions 1 - 6 and at least 4 from questions 7 - 12. On the front of your paper indicate which 10 problem you wish to have graded.

Problem 1. Let $f \in C^2(\mathbf{R})$ be a real valued function that is uniformly bounded on $\mathbf{R}$. Prove that there exists a point $c \in \mathbf{R}$ such that $f''(c) = 0$.

Problem 2. Let μ be a Borel probability measure on $[0,1]$ that has no atoms (this means that $\mu(\{t\}) = 0$ for any $t \in [0,1]$). Let also $\mu_1, \mu_2, \ldots$ be Borel probability measures on $[0,1]$. Assume that μ_n converges to μ in the weak* topology. Denote $F(t) := \mu([0,t])$ and $F_n(t) := \mu_n([0,t])$ for each $n \geq 1$ and $t \in [0,1]$. Prove that F_n converges uniformly to F.

Problem 3. (*a*) Let f be a positive continuous function on $\mathbf{R}$ such that $\lim\limits_{|t|\to\infty} f(t) = 0$. Show that the set $\{hf \mid h \in L^1(\mathbf{R}, m), \|h\|_1 \leq K\}$ is a closed nowhere dense set in $L^1(\mathbf{R}, m)$, for any $K \geq 1$ (m denotes the Lebesgue measure on $\mathbf{R}$).

(*b*) Let $\{f_n\}_n$ be a sequence of positive continuous functions on $\mathbf{R}$ such that for each n we have $\lim\limits_{|t|\to\infty} f_n(t) = 0$. Show that there exists $g \in L^1(\mathbf{R}, m)$ such that $g/f_n \notin L^1(\mathbf{R}, m)$ $\forall n$.

Problem 4. Let $\mathcal{V}$ be the subspace of $L^\infty([0,1], \mu)$ (where μ is the Lebesgue measure on $[0,1]$) defined by

$$\mathcal{V} = \{f \in L^\infty([0,1], \mu) \mid \lim_{n\to\infty} n \int_{[0,1/n]} f \mathrm{d}\mu \text{ exists}\}$$

(*a*) Prove that there exists $\varphi \in L^\infty([0,1], \mu)^*$ (i.e., a continuous functional on $L^\infty([0,1], \mu)$) such that $\varphi(f) = \lim\limits_{n\to\infty} n \int_{[0,1/n]} f \mathrm{d}\mu$ for every $f \in \mathcal{V}$.

(*b*) Show that, given any $\varphi \in L^\infty([0,1], \mu)^*$ satisfying the condition in (*a*) above, there exists no $g \in L^1([0,1], \mu)$ such that $\varphi(f) = \int fg \, \mathrm{d}\mu$ for all $f \in L^\infty([0,1], \mu)$.

Problem 5. (*a*) Prove that $L^p([0,1], \mu)$ are separable Banach spaces for $1 \leq p < \infty$ but $L^\infty([0,1], \mu)$ is not (where μ is the Lebesgue measure on $[0,1]$).

(*b*) Prove that there exists no linear bounded surjective map $T : L^p([0,1], \mu) \to L^1([0,1], \mu)$ if $p > 1$.

Problem 6. Let $\mathcal{H}$ be a Hilbert space and $\{\xi_n\}_n$ a sequence of vectors in $\mathcal{H}$ such that $\|\xi_n\| = 1$ for all n.

(*a*) Show that if $\{\xi_n\}_n$ converges weakly to a vector $\xi \in \mathcal{H}$ with $\|\xi\| = 1$, then $\lim\limits_{n\to\infty} \|\xi_n - \xi\| = 0$.

(b) Show that if $\lim_{n,m\to\infty} \|\xi_n + \xi_m\| = 2$, then there exists a vector $\xi \in \mathcal{H}$ such that $\lim_{n\to\infty} \|\xi_n - \xi\| = 0$.

Problem 7. Let $f : \mathbf{C} \to \mathbf{C}$ be entire non-constant, and let us set

$$T(r) = \frac{1}{2\pi} \int_0^{2\pi} \log_+ \left| f(re^{i\varphi}) \right| \, d\varphi.$$

Here $\log_+ s = \max(\log s, 0)$. Show that $T(r) \to \infty$ as $r \to \infty$.

Problem 8. Show that

$$\sin z - z \cos z = \frac{z^3}{3} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right), \quad z \in \mathbf{C},$$

where $(\lambda_n)_{n\geq 1}$ is a sequence in $\mathbf{C}$, $\lambda_n \neq 0$ for all n, such that

$$\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|^2} < \infty.$$

Problem 9. Let $\mathbf{D} = \{z \in \mathbf{C};\ |z| < 1\}$ and let $\mathcal{A}(\mathbf{D})$ be the space of functions holomorphic in $\mathbf{D}$ and continuous in $\overline{\mathbf{D}}$. Let

$$\mathcal{U} = \{f \in \mathcal{A}(\mathbf{D});\ |f(z)| = 1 \quad \text{for all } z \in \partial\mathbf{D}\}.$$

Show that $f \in \mathcal{U}$ if and only if f is a finite Blaschke product,

$$f(z) = \lambda \prod_{j=1}^{N} \frac{z - a_j}{1 - \overline{a_j} z},$$

for some $a_j \in \mathbf{D}$, $1 \leq j \leq N < \infty$ and $|\lambda| = 1$.

Problem 10. For $a > 0$, $b > 0$, evaluate the integral

$$\int_0^{\infty} \frac{\log x}{(x+a)^2 + b^2} \, dx.$$

Problem 11. Let $u \in C^{\infty}(\mathbf{R})$ be smooth 2π-periodic. Show that there exists a bounded holomorphic function f_+ in the upper half-plane $\operatorname{Im} z > 0$ and a bounded holomorphic function f_- in the lower half-plane $\operatorname{Im} z < 0$, such that

$$u(x) = \lim_{\varepsilon\to 0+} \left(f_+(x + i\varepsilon) - f_-(x - i\varepsilon)\right), \quad x \in \mathbf{R}.$$

Problem 12. Let $\mathcal{H}$ be the vector space of entire functions $f : \mathbf{C} \to \mathbf{C}$ such that

$$\int_{\mathbf{C}} |f(z)|^2 \, d\mu(z) < \infty.$$

Here $d\mu(z) = e^{-|z|^2} \, d\lambda(z)$, where $d\lambda(z)$ is the Lebesgue measure on $\mathbf{C}$.

1. Show that $\mathcal{H}$ is a closed subspace of $L^2(\mathbf{C}, d\mu)$.
2. Show that for all $f \in \mathcal{H}$, we have

$$f(z) = \frac{1}{\pi} \int_{\mathbf{C}} f(w) e^{z\overline{w}} \, d\mu(w), \quad z \in \mathbf{C}.$$

Hint for 2): Show that the normalized monomials

$$e_n(z) = \frac{1}{(\pi n!)^{1/2}} z^n, \quad n = 0, 1, \ldots$$

form an orthonormal basis of $\mathcal{H}$.