Analysis Qualifying Exam, September 17, 2020

Rules: Solve no more that ten problems. You must demonstrate adequate knowledge of both real analysis (Problems 1–6) and complex analysis (Problems 7–12). Parts of a problem may not carry equal weight.

Problem 1. (a) Suppose $f : [0,1] \times [0,\infty) \to [0,1]$ is continuous. Prove that $F : [0,1] \to [0,1]$ defined by

$$F(x) = \limsup_{y\to\infty} f(x,y)$$

is Borel measurable.
(b) Show that for any Borel set $E \subseteq [0,1]$ there is a choice of continuous function $f : [0,1] \times \mathbb{R} \to [0,1]$ so that F agrees almost everywhere with the indicator function of E.

Problem 2. Show that there is a constant $c \in \mathbb{R}$ so that

$$\lim_{n\to\infty} \int_0^1 f(x)\, \cos\bigl(\sin(n\pi x)\bigr)\, dx = c \int_0^1 f(x)\, dx$$

for every $f \in L^1([0,1])$. The limit is taken over those $n \in \mathbb{N}$.

Problem 3. Let $d\mu_n$ be a sequence of probability measures on $[0,1]$ so that

$$\int f(x)\, d\mu_n(x)$$

converges for every continuous function $f : [0,1] \to \mathbb{R}$.
(a) Show that

$$\iint g(x,y)\, d\mu_n(x)\, d\mu_n(y)$$

converges for every continuous $g : [0,1]^2 \to \mathbb{R}$.
(b) Show by example that under the above hypotheses, it is possible that

$$\iint_{0\le x\le y\le 1} d\mu_n(x)\, d\mu_n(y)$$

does not converge.

Problem 4. Let X be a separable Banach space over $\mathbb{R}$ and let $F : X \to \mathbb{R}$ be norm-continuous and convex. Suppose now that a sequence x_n in X converges weakly to $x \in X$. Show that

$$F(x) \leq \sup_n F(x_n).$$

Problem 5. Suppose $f \in L^1([0,1])$ has the property that

$$\int_E |f(x)|\, dx \leq \sqrt{|E|}, \qquad (*)$$

for every Borel $E \subseteq [0,1]$. Here $|E|$ denotes the Lebesgue measure of E.

(a) Show that $f \in L^p([0,1])$ for all $p > 2$.

(b) Give an example of an f satisfying $(*)$ that is not it $L^2([0,1])$.

Problem 6. Prove that the following inequality is valid for all odd C^1 functions $f : [-1,1] \to \mathbb{R}$:

$$\int_{-1}^{1} |f(x)|^2\, dx \leq \int_{-1}^{1} |f'(x)|^2\, dx$$

By odd, we mean that $f(-x) = -f(x)$.

Problem 7. Let $\Delta_j = \{z : |z - a_j| \leq r_j\}$, $1 \leq j \leq n$, be a collection of disjoint closed disks, with radii $r_j \geq 0$, all contained in the open unit disk $\mathbb{D}$ of the complex plane. Let $\Omega = \mathbb{D} \setminus (\cup_j \Delta_j)$ and let $u : \Omega \to \mathbb{R}$ be harmonic. Prove that there exist real numbers $c_1, \ldots, c_n$ such that

$$u(z) - \sum_{j=1}^{n} c_j \log |z - a_j|$$

is the real part of a (single valued) analytic function on Ω. Show also that the choice of $c_1, \ldots, c_n$ is unique.

Problem 8. Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic and satisfy $f(\frac{1}{2}) = f(-\frac{1}{2}) = 0$. Show that

$$\big|f(0)\big| \leq \tfrac{1}{4}.$$

Problem 9. Consider the following region in the complex plane:

$$\Omega = \{x + iy : 0 < x < \infty \text{ and } 0 < y < \tfrac{1}{x}\}.$$

Exhibit an explicit conformal mapping f of Ω onto $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Problem 10. Let $K \subset \mathbb{C}$ be a compact set of positive area but empty interior and define a function $F : \mathbb{C} \to \mathbb{C}$ via

$$F(z) = \iint_K \frac{1}{w - z} d\mu(w),$$

where $d\mu$ denotes (planar) Lebesgue measure on $\mathbb{C}$.

(a) Prove that $F(z)$ is bounded and continuous on $\mathbb{C}$ and analytic on $\mathbb{C} \setminus K$.

(b) Prove that $\{F(z) : z \in \mathbb{C}\} = \{F(z) : z \in K\}$.

Hint: If $a \in F(\mathbb{C}) \setminus F(K)$ and $F^{-1}(a) = \{z_1, ..., z_n\} \subset \mathbb{C} \setminus K$, then the argument principle can be applied to $G(z) = \frac{F(z)-a}{\prod_j (z-z_j)}$ to get a contradiction.

Problem 11. Let $\{f_n\}$ be a sequence of analytic functions on a (connected) domain Ω such that $|f_n(z)| \leq 1$ for all n and all $z \in \Omega$. Suppose the sequence $\{f_n(z)\}$ converges for infinitly many z in a compact subset K of Ω. Prove $\{f_n(z)\}$ converges for all $z \in \Omega$.

Problem 12. Let $\Omega = \{z \in \mathbb{C} : -2 < \operatorname{Im} z < 2\}$. Show that there is a finite constant C so that

$$|f(0)|^2 \leq C \int_{-\infty}^{\infty} \left[|f(x + i)|^2 + |f(x - i)|^2\right] dx$$

for every holomorphic $f : \Omega \to \mathbb{D}$ for which the right-hand side is finite.