

ANALYSIS QUAL: JUNE 18, 2020

Please be reminded that to pass the exam you need to show mastery of both real and complex analysis. Please choose at most 10 questions to answer, including at least 4 from problems 1–6 and 4 from problems 7–12. On the front of your paper indicate which 10 problems you wish to have graded.

Problem 1. Assume $f \in C_c^\infty(\mathbb{R})$ satisfies

$$\int_{\mathbb{R}} e^{-tx^2} f(x) dx = 0 \quad \text{for any } t \geq 0.$$

Show that $f(x) = -f(-x)$ for any $x \in \mathbb{R}$.

Problem 2. Assume $f_n : \mathbb{R} \rightarrow \mathbb{R}$ is a sequence of differentiable functions satisfying

$$\int_{\mathbb{R}} |f_n(x)| dx \leq 1 \quad \text{and} \quad \int_{\mathbb{R}} |f'_n(x)| dx \leq 1.$$

Assume also that for any $\varepsilon > 0$ there exists $R(\varepsilon) > 0$ such that

$$\sup_n \int_{|x| \geq R(\varepsilon)} |f_n(x)| dx < \varepsilon.$$

Show that there exists a subsequence of $\{f_n\}$ that converges in $L^1(\mathbb{R})$.

Problem 3. Prove that $L^\infty(\mathbb{R}^n) \cap L^3(\mathbb{R}^n)$ is a Borel subset of $L^3(\mathbb{R}^n)$.

Problem 4. Fix $f \in L^1(\mathbb{R})$. Show that

$$\lim_{n \rightarrow \infty} \int_0^2 f(x) \sin(x^n) dx = 0.$$

Problem 5. Rigorously determine the infimum of

$$\int_{-1}^1 |P(x) - |x||^2 dx$$

over all choices of polynomials $P \in \mathbb{R}[x]$ of degree not exceeding three.

Problem 6. Let us define a sequence of linear functionals on $L^\infty(\mathbb{R})$ as follows:

$$L_n(f) = \frac{1}{n!} \int_0^\infty x^n e^{-x} f(x) dx.$$

- (a) Prove that no subsequence of this sequence converges weak-*
- (b) Explain why this does not contradict the Banach–Alaoglu Theorem.

Problem 7. Let \mathcal{F}_M be the set of functions holomorphic on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and continuous on $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ that satisfy

$$\int_0^{2\pi} |f(e^{it})| dt \leq M < \infty.$$

Show that every sequence $\{f_n\}$ in \mathcal{F}_M contains a subsequence that converges uniformly on compact subsets of \mathbb{D} .

Problem 8. For each $z \in \mathbb{C}$, let

$$F(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(z/2)^{2n}}{(n!)^2}.$$

(a) Show that F is an entire function and satisfies $|F(z)| \leq e^{|z|}$.

(b) Show that there is an infinite collection of numbers $a_n \in \mathbb{C}$, so that

$$F(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{a_n^2}\right)$$

and the product converges uniformly on compact subsets of \mathbb{C} .

Problem 9. Let $f \in L^1(\mathbb{C}) \cap C^1(\mathbb{C})$. Show that the integral

$$u(z) = -\frac{1}{2\pi} \iint_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z} d\lambda(\zeta), \quad z \in \mathbb{C},$$

defines a C^1 function on the whole complex plane that satisfies

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)u(x + iy) = f(x + iy).$$

In this problem, $d\lambda(\zeta)$ denotes (planar) Lebesgue measure on \mathbb{C} and C^1 is meant in the real-variables sense.

Problem 10. Evaluate the improper Riemann integral

$$\int_0^{\infty} \frac{x^2 - 1}{x^2 + 1} \frac{\sin x}{x} dx.$$

Justify all manipulations.

Problem 11. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and let $K \subsetneq \mathbb{T}$ be a compact proper subset.

(a) Show that there is a sequence of polynomials $P_n(z)$ so that $P_n(z) \rightarrow \bar{z}$ uniformly on K .

(b) Show that there is *no* sequence of polynomials $P_n(z)$ for which $P_n(z) \rightarrow \bar{z}$ uniformly on \mathbb{T} .

Problem 12. Let u be a continuous subharmonic function on \mathbb{C} that satisfies

$$\limsup_{|z| \rightarrow \infty} \frac{u(z)}{\log |z|} \leq 0.$$

Show that u is constant on \mathbb{C} .