Basic Examination F02

1. Let K be a compact subset and F be a closed subset in the metric space X. Suppose $K \cap F = \phi$. Prove that

$$0 < \inf\{d(x,y) : x \in K, y \in F\}.$$

2. Show why the Least Upper Bound Property (every set bounded above has a least upper bound) implies the Cauchy Completeness Property (every Cauchy sequence has a limit) of the real numbers.

3. Show that there is a subset of the real numbers which is not the countable intersection of open subsets.

4. By integrating the series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 \cdots$$

prove that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots$. Justify carefully all the steps (especially taking the limit as $x \to 1$ from below).

5. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ has partial derivatives at every point bounded by $A > 0$.

   (a) Show that there is an $M > 0$ such that

$$|f((x,y)) - f((x_1,y_1))| \le M((x-x_1)^2 + (y-y_1)^2)^{1/2}.$$

   (b) What is the smallest value of M (in terms of A) for which this always works?

   (c) Give an example where that value of M makes the inequality an equality.

6. Suppose $F : \mathbb{R}^3 \to \mathbb{R}^2$ is continuously differentiable. Suppose for some $v_0 \in \mathbb{R}^3$ and $x_0 \in \mathbb{R}^2$ that $F(v_0) = x_0$ and $F'(v_0) : \mathbb{R}^3 \to \mathbb{R}^2$ is onto. Show that there is a continuously differentiable function γ, $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ for some $\varepsilon > 0$, such that

   (i) $\gamma'(0) \ne \vec{0} \in \mathbb{R}^3$, and

   (ii) $F(\gamma(t)) = x_0$ for all $t \in (-\varepsilon, \varepsilon)$.

7. Let $T: V \to W$ be a linear transformation of finite dimensional real vector spaces. Define the transpose of T and then prove both of the following:

   i. $(\{\text{im}\}(T))^0 = \{\ker(T^t)\}$ where $(\{\text{im}\}(T))^0$ is the annihilator of $\{\text{im}\}(T)$, the image (range) of T, and $\{\ker(T^t)\}$ is the kernel (null space) of T^t.

   ii. $\{\text{rank}(T)\} = \{\text{rank}\}(T^t)$, where the rank of a linear transformation is the dimension of its image.

8. Let T be the rotation of an angle 60^0 counterclockwise about the origin in the plane perpendicular to $(1, 1, 2)$ in $\{\mathbf{R}\}^3$.

   i. Find the matrix representation of T in the standard basis. Find all eigenvalues and eigenspaces of T.

   ii. What are the eigenvalues and eigenspaces of T if $\{\mathbf{R}\}^3$ is replaced by $\{\mathbf{C}\}^3$.

   [You do not have to multiply any matrices out but must compute any inverses.]

9. Let V be a complex inner product space. State and prove the Cauchy-Schwarz inequality.

10. Let A be an $n \times n$ complex matrix satisfying $A^*A = AA^*$ where A^* is the adjoint of A. Let $V = \{\mathbf{C}\}^{\{n\times 1\}}$, the $n \times 1$ complex column matrices, be an inner product space under the dot product. View $A: V \to V$ as a linear map. Prove that there exists an orthonormal basis of V consisting of eigenvectors of A, i.e., prove this form of the Spectral Theorem for normal operators.