

1. Prove that the closed interval $[0, 1]$ is connected.
2. Show that the set Q of rational numbers in \mathbb{R} is not expressible as the intersection of a countable collection of open subsets of \mathbb{R} .
3. Suppose that X is a compact metric space (in the covering sense of the word compact). Prove that every sequence $\{x_n : x_n \in X, n = 1, 2, 3, \dots\}$ has a convergent subsequence. [Prove this directly. Do not just quote a theorem.]
4. (a) Define *uniform continuity* of a function $F : X \rightarrow \mathbb{R}$, X a metric space.
(b) Prove that a function $f : (0, 1) \rightarrow \mathbb{R}$ is the restriction to $(0, 1)$ of a continuous function $F : [0, 1] \rightarrow \mathbb{R}$ if and only if f is uniformly continuous on $(0, 1)$.
5. State some reasonable conditions under which a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right)$$

everywhere on \mathbb{R}^2 and prove this equality under the conditions you give.

6. Suppose $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a continuously differentiable function with $\text{grad } f \neq \vec{0}$ at $\vec{0}$ ($\vec{0} = (0, 0, 0)$ in \mathbb{R}^3). Show that there are two other continuously differentiable functions $g : \mathbb{R}^3 \rightarrow \mathbb{R}$, $h : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that the function

$$(x, y, z) \rightarrow (f(x, y, z), g(x, y, z), h(x, y, z))$$

from \mathbb{R}^3 to \mathbb{R}^3 is one-to-one on some neighborhood of $\vec{0}$.

7. Suppose $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is continuously differentiable and that the Jacobian matrix of F is everywhere nonsingular. Suppose also that $F(\vec{0}) = \vec{0}$ and that $\|F((x, y))\| \geq 1$ for all (x, y) with $\|(x, y)\| = 1$.

Prove that $F(\{(x, y) : \|(x, y)\| < 1\}) \supset \{(x, y) : \|(x, y)\| < 1\}$.

(Hint: Show, with $U = \{(x, y) : \|(x, y)\| < 1\}$, that $F(U) \cap U$ is both open and closed in U).

8. Let V be a finite dimensional real vector space. Let $W \subset V$ be a subspace and $W^\circ := \{f : V \rightarrow F \text{ linear} \mid f = 0 \text{ on } W\}$. Prove that

$$\dim(V) = \dim(W) + \dim(W^\circ).$$

9. Find the matrix representation in the standard basis for either rotation by an angle θ in the plane perpendicular to the subspace spanned by the vectors $(1,1,1,1)$ and $(1,1,1,0)$ in \mathbf{R}^4 .

[You do not have to multiply the matrices out but must compute any inverses.]

10. Let V be a complex inner product space and W a finite dimensional subspace. Let $v \in V$. Prove that there exists a unique vector $v_W \in W$ such that

$$\|v - v_W\| \leq \|v - w\|$$

for all $w \in W$. Deduce that equality holds if and only if $w = v_W$.

11. Let V be a finite dimensional real inner product space and $T, S : V \rightarrow V$ two commuting hermitian linear operators. Show that there exists an orthonormal basis for V consisting of vectors that are simultaneously eigenvectors of T and S .