Basic Exam (S04)

In several problems you will need the usual "norm" terminology. If $V$ is a real vector space, then a norm on $V$ is a map $\|\ \|: V \to [0,\infty)$ such that $\|v+w\| \le \|v\| + \|w\|$, $\|cv\| = |c|\|v\|$, and $\|v\| = 0$ if and only if $v = 0$. Each norm determines a metric $d$ on $V$ via the relation $d(v,w) = \|v-w\|$. The Euclidean norm (also called the "inner product" norm) on $\mathbb{R}^n$ is given by

$$\left\|\sum_{k=1}^{n} x_k e_k\right\|_2 = [\sum_{k=1}^{n} |x_k|^2]^{1/2}.$$

where $e_k$ is the usual vector basis. Given a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ we define

$$\|T\| = \sup\{\|T(x)\|_2 : \|x\|_2 \le 1\}.$$

For all $x$, $\|T(x)\| \le \|T\|\|x\|$.

1. Let $\mathcal{S}$ denote the set of sequences $a = (a_1, a_2, \ldots)$, with $a_k = 0$ or 1. Show that the mapping $\theta : \mathcal{S} \to \mathbb{R}$ defined by

$$\theta((a_1, a_2, \ldots)) = \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots$$

is an injection. Include an explanation of why the infinite series converges. Hint: if $a \ne b$, you may assume that

$$\begin{aligned} a &= (a_1, \ldots, a_{n-1}, 0, a_{n+1}, \ldots). \\ b &= (a_1, \ldots, a_{n-1}, 1, b_{n+1}, \ldots) \end{aligned}$$

2. Is $f(x) = \sqrt{x}$ uniformly continuous on $[0,\infty)$? Prove your assertion.

3. a) Carefully define when a function $f$ on $[0,1]$ is Riemann integrable.
b) Show that if $f_n$ are Riemann integrable functions on $[0,1]$ and $f_n$ converges to $f$ uniformly, then $f$ is Riemann integrable.

4. Are there infinite compact subsets of $\mathbb{Q}$? Prove your assertion.

5. Suppose that $G$ is an open set in $\mathbb{R}^n$, $f : G \to \mathbb{R}^m$ is a function, and that $x_0 \in G$.
a) Carefully define what is meant by $f'(x_0) : \mathbb{R}^n \to \mathbb{R}^m$.
b) Suppose that $I$ is a line segment in $G$ such that $f'(x)$ is defined for all $x \in I$. Show that if $f$ is differentiable at all the points of $I$, then for some point $c$ in $I$

$$\|f(q) - f(p)\|_2 \le \|f'(c)\|\,\|q-p\|_2\,.$$

Hint: let $w$ be a unit vector with $\|f(q) - f(p)\|_2 = (f(q) - f(p)) \cdot w$.

6. Let $\| \ \|$ be any norm on $\mathbb{R}^n$.

a) Prove that there exists a constant $d$ with $\|x\| \leq d\,\|x\|_2$ for all $x \in \mathbb{R}^n$, and use this to show that $N(x) = \|x\|$ is continuous in the usual topology on $\mathbb{R}^n$.

b) Prove that there exists a constant $c$ with $\|x\| \geq c\,\|x\|_2$ (Hint: use the fact that $N$ is continuous on the sphere $\{x : \|x\|_2 = 1\}$).

c) Show that if $L$ is an $n$-dimensional subspace of an arbitrary normed vector space $V$, then $L$ is closed.

7. Let $V$ be a finite dimensional real vector space. Let $W_1, W_2 \subset V$ be subspaces. Show both of the following:

a) $W_1^0 \cap W_2^0 = (W_1 + W_2)^0$

b) $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$

[Note: $W_i^0$ is the annihilator of $W_i$.]

8. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be a rotation about the axis $(1, 0 - 1)$ by an angle of $30^o$ (you can use either orientation).

a) Find the matrix representation $A \in \mathbf{M}_3(\mathbf{R})$ of $T$ in the standard basis. (You do not have to multiply out matrices but must evaluate inverses.)

b) Find all the eigenvalues of $A \in \mathbf{M}_3(\mathbf{R})$.

c) Find all the eigenvalues of $A \in \mathbf{M}_3(\mathbf{C})$.

9. Let $V$ be a finite dimensional real inner product space under $( \ , \ )$ and $T : V \to V$ a linear operator. Show the following are equivalent:

a) $(Tx, Ty) = (x, y)$ for all $x, y \in V$.

b) $||T(x)|| = ||x||$ for all $x \in V$.

c) $T^*T = Id_V$, where $T^*$ is the adjoint of $T$.

d) $TT^* = Id_V$.

10. Let $T$ be a real symmetric matrix. Show that $T$ is similar to a diagonal matrix.

[You cannot use the Spectral Theorem.]