Note: Throughout this exam, $M_n(\mathbb{C})$ denotes the set of $n \times n$ matrices with complex entries.

Linear Algebra.

1. Given $n \geq 1$, let $\text{tr} : M_n(\mathbb{C}) \to \mathbb{C}$ denote the trace of a matrix:

$$\text{tr}(A) = \sum_{k=1}^{n} A_{k,k}.$$

   (a) Determine a basis for the kernel (or null-space) of tr.

   (b) For $X \in M_n(C)$, show that $\text{tr}(X) = 0$ if and only if there exists an integer m and matrices $A_1, \ldots, A_m, B_1, \ldots, B_m \in M_n(\mathbb{C})$ so that

$$X = \sum_{i=j}^{m} A_j B_j - B_j A_j$$

2. Let V be a finite-dimensional vector space, and let V^* denote the dual space; that is, the space of linear maps $\phi : V \to \mathbb{C}$. For a set $W \subset V$, let

$$W^{\perp} = \{\phi \in V^* : \phi(w) = 0 \; \forall w \in W\}.$$

   For a subset $U \subset V^*$, let

$$^{\perp}U = \{v \in V : \phi(v) = 0 \; \forall \phi \in U\}.$$

   (a) Show that for any subset $W \subset V$, $^{\perp}(W^{\perp}) = \text{span}(W)$.
   Recall that the span of a set of vectors is the smallest vector sub-space that contains these vectors.

   (b) Let $W \subset V$ be a linear subspace. Give an explicit isomorphism between $(V/W)^*$ and $W^{\perp}$. Show that it is an isomorphism.

3. Let A be a Hermitian-symmetric $n \times n$ complex matrix. Show that if $\langle Av, v \rangle \geq 0$ for all $v \in \mathbb{C}^n$, then there exists an $n \times n$ matrix T so that $A = T^* T$.

4. Let $\mathcal{A} = M_n(\mathbb{C})$ denote the set of all $n \times n$ matrices with complex entries. We say that $\mathcal{I} \subseteq \mathcal{A}$ is a *two-sided ideal* in $\mathcal{A}$ if

   (i) for all $A, B \in \mathcal{I}$, $A + B \in \mathcal{I}$
   (ii) for all $A \in \mathcal{I}$ and $B \in \mathcal{A}$, AB and BA belong to $\mathcal{I}$

   Show that the only two-sided ideals in $\mathcal{A}$ are $\{0\}$ and $\mathcal{A}$ itself.

Analysis.

1. For a subset $X \subset \mathbb{R}$, we say that X is *algebraic*, if there exists a family $\mathcal{F}$ of polynomials with rational coefficients, so that $x \in X$ if and only if $p(x) = 0$ for some $p \in \mathcal{F}$.

   (a) Show that the set $\mathbb{Q}$ of rational numbers is algebraic.

   (b) Show that the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is not algebraic.

2. Let X be the set of all infinite sequences $\{\sigma_n\}_{n=1}^{\infty}$ of 1's and 0's endowed with the metric

$$\operatorname{dist}\left(\{\sigma_n\}_{n=1}^{\infty}, \{\sigma'_n\}_{n=1}^{\infty}\right) = \sum_{n=1}^{\infty} \frac{1}{2^n} |\sigma_n - \sigma'_n|.$$

   Give a direct proof that every infinite subset of X has an accumulation point.

3. Let X, Y be two topological spaces. We say that a continuous function $f : X \to Y$ is *proper* if $f^{-1}(K)$ is compact for any compact set $K \subset Y$.

   (a) Give an example of a function that is proper but not a homeomorphism.

   (b) Give an example of a function that is continuous but not proper.

   (c) Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^1 (that is, has a continuous derivative) and

$$|f'(x)| \geq 1 \qquad \text{for all } x \in \mathbb{R}.$$

   Show that f is proper.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^1 (i.e., continuously differentiable). Show that

$$\lim_{n \to \infty} \sum_{j=1}^{n} |f(\tfrac{j-1}{n}) - f(\tfrac{j}{n})|$$

   is equal to

$$\int_0^1 |f'(t)|\, dt.$$

5. (a) Suppose

$$\lim_{n \to \infty} a_n = A$$

   Show that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} a_n = A$$

   (b) Show by example that the converse is false.

6. Consider the set of $f : [0,1] \to \mathbb{R}$ that obey

$$|f(x) - f(y)| \le |x - y| \qquad \text{and} \qquad \int_0^1 f(x)\,dx = 1.$$

Show that this is a compact subset of $C([0,1])$.

7. Let us make $M_n(\mathbb{C})$ into a metric space in the following fashion:

$$\operatorname{dist}(A, B) = \left\{ \sum_{i,j} |A_{i,j} - B_{i,j}|^2 \right\}^{1/2}$$

(which is just the usual metric on $\mathbb{R}^{n^2}$).

(a) Suppose $F : \mathbb{R} \to M_n(\mathbb{C})$ is continuous. Show that the set

$$\{x \in \mathbb{R} : F(x) \text{ is invertible}\}$$

is open (in the usual topology on $\mathbb{R}$).

(b) Show that on the set given above, $x \mapsto [F(x)]^{-1}$ is continuous.

8. Let (X, d) be a metric space. Prove that the following are equivalent:

(a) There is a countable dense set.

(b) There is a countable basis for the topology.

Recall that a collection of open sets $\mathcal{U}$ is called a basis if every open set can be written as a union of elements of $\mathcal{U}$.