BASIC QUAL WINTER 2006
(February 18, 2006)

Problem 1. Show that for each $\epsilon > 0$ there exists a sequence of intervals (I_n) with the properties

$$\bigcup_{n=1}^{\infty} I_n \supset \mathbb{Q} \quad \text{and} \quad \sum_{n=1}^{\infty} |I_n| < \epsilon.$$

Problem 2. Let $(a_n)_{n\geq 1}$ be a decreasing sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n = \infty$. Under what condition(s) is the function

$$f(x) = \sum_{n=1}^{\infty} (-1)^n a_n \, x^n$$

well-defined and left-continuous at $x = 1$? Carefully prove your assertion.

Problem 3. Consider a function $f\colon [a,b] \to \mathbb{R}$ which is twice continuously differentiable (including the endpoints). Let $a = x_0 < x_1 < \cdots < x_n = b$ be the uniform partition of $[a,b]$, i.e., $x_{i+1} - x_i = (b-a)/n$ for all $0 \leq i < n$. Show that there exists M such that for all $n \geq 1$,

$$\left| \frac{1}{n}\Big(\frac{1}{2}f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\Big) - \int_a^b f(x)\mathrm{d}x \right| \leq \frac{M}{n^2}.$$

[Recall that the sum is an approximation of the integral in the Trapezoid Rule. It may be instructive to first solve the problem for $n = 1$ and then address the general case.]

Problem 4. Consider a decreasing sequence of continuous functions $f_n\colon [0,1] \to \mathbb{R}$ obeying the uniform bound $|f_n| \leq M$ for some $M \in (0,1)$. Suppose the point-wise limit $f(x) = \lim_{n\to\infty} f_n(x)$ is continuous on $[0,1]$. Prove that $f_n \to f$ uniformly on $[0,1]$.
[You may use without proof that $[0,1]$ is compact as well as sequentially compact.]

Problem 5. Consider a function $f(x,y)$ which is twice continuously differentiable. Suppose that f has its unique minimum at $(x,y) = (0,0)$. Carefully prove that then at $(0,0)$,

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} \geq \Big(\frac{\partial^2 f}{\partial x \partial y}\Big)^2$$

[You may use without proof that the mixed partials are equal for C^2 functions.]

Problem 6. Let $-\infty < a < b < \infty$. Prove that a continuous function $f\colon [a,b] \to \mathbb{R}$ attains all values in $[f(a), f(b)]$.

Problem 7. Let V be a complex inner product space and $v, w \in V$. Prove the Cauchy-Schwarz inequality

$$|(v, w)| \leq |v||w|.$$

Problem 8. Let $T\colon V \rightarrow W$ be a linear transformation of finite dimensional real inner product spaces. Show that there exists a unique linear transformation $T^t\colon W \rightarrow V$ such that

$$\langle T(v), w\rangle_W = \langle v, T^t(w)\rangle_V \text{ for all } v \in V \text{ and } w \in W$$

where $\langle\ ,\ \rangle_X$ is the inner product on $X = V$ or W.

Problem 9. Let $A \in \mathbb{M}_3(\mathbb{R})$ be invertible and satisfy $A = A^t$ and $\det A = 1$. Prove that A has one as an eigenvalue.

Problem 10. Let $T\colon V \rightarrow V$ be a linear operator on a finite dimensional complex inner product space. Show that there exists an ordered orthonormal basis for V such that the matrix representation A of T in this basis is upper triangular, i.e, $A = (a_{ij})$ with $a_{ij} = 0$ if $j < i$.
[You cannot use canonical form theorems without proof.]