Basic Exam, Spring 2007

1. Let A be a real $m \times n$ matrix, $m > n$, whose columns are linearly independent and $\mathbf{b} \in \mathrm{R}^m$. Show that the vector $\mathbf{x}^* \in \mathrm{R}^n$ that minimizes the functional

$$g(\mathbf{x}) = ||A\mathbf{x} - \mathbf{b}||_2^2$$

is given by the solution of the normal equations

$$A^t A\mathbf{x} = A^t \mathbf{b}.$$

Here $||\mathbf{z}||_2^2 = \langle \mathbf{z}, \mathbf{z} \rangle = \sum_i z_i^2$.

2. Let V, W, Z be n-dimensional vector spaces and $T : V \to W$ and $U : W \to Z$ be linear transformations. Prove that if the composite transformation $UT : V \to Z$ is invertible, then both T and U are invertible. (Do not use determinants in your proof!)

3. Consider the space of infinite sequences of real numbers

$$\mathcal{S} = \{(a_0, a_1, a_2, ...) : a_n \in \mathrm{R}, n = 0, 1, 2, ...\}$$

endowed with the standard operations of addition and scalar multiplication:

$$(a_0, a_1, ...) + (b_0, b_1, ...) = (a_0 + b_0, a_1 + b_1, ...); \quad c(a_0, a_1, ...) = (ca_0, ca_1, ...), \quad c \in \mathrm{R}.$$

For each pair of real numbers A and B, prove that the set of solutions $(x_0, x_1, x_2, ...)$ of the linear recursion

$$x_{n+2} = Ax_{n+1} + Bx_n, \quad n = 0, 1, 2, ...$$

is a linear subspace of $\mathcal{S}$ of dimension 2.

4. Suppose that A is a symmetric $n \times n$ real matrix with distinct eigenvalues $\lambda_1, ..., \lambda_l$, $(l \leq n)$. Find the sets

$$X = \left\{ \mathbf{x} \in \mathrm{R}^n : \lim_{k \to \infty} (\mathbf{x}^t A^{2k} \mathbf{x})^{1/k} \text{ exists} \right\}$$

and

$$L = \left\{ \lim_{k \to \infty} (\mathbf{x}^t A^{2k} \mathbf{x})^{1/k} : x \in X \right\},$$

where R^n is identified with the set of real column vectors, and $\mathbf{x}^t$ denotes the transpose of $\mathbf{x}$.

5. Let T be a normal linear operator on a finite dimensional complex inner product linear space V. Prove that if $\mathbf{v}$ is an eigenvector of T, then $\mathbf{v}$ is also an eigenvector of its adjoint T^*.

6. Consider the integral equation

$$(*) \qquad y(t) = y_0 + \int_0^t f(s, y(s))ds$$

where $f(t, y)$ is continuous on $[0, T] \times \mathrm{R}$ and is Lipschitz in y with Lipschitz constant K. Assume that you have shown that the iterates defined by

$$y^n(t) = y_0 + \int_0^t f(s, y^{n-1}(s))ds, \quad y^0(t) \equiv y_0$$

converge uniformly to a solution $y(t)$ of (*). Show that if $Y(t)$ is a solution of (*) and satisfies $|Y(t) - y_0| \le C$ for some constant C and all $t \in [0, T]$, then $Y(t) \equiv y(t)$ on $[0, T]$.

7. Let $f : \mathrm{R} \to \mathrm{R}$ be a twice continuously differentiable function with f'' uniformly bounded, and with a simple root at x^* (i.e., $f(x^*) = 0, f'(x^*) \ne 0$). Consider the fixed point iteration

$$x_n = F(x_{n-1}) \qquad \text{where} \qquad F(x) = x - \frac{f(x)}{f'(x)}.$$

Show that if x_0 is sufficiently close to x^*, then there exists a constant C so that for all n,

$$|x_n - x^*| \le C|x_{n-1} - x^*|^2.$$

8. Suppose the functions f_n are twice continuously differentiable on $[0, 1]$ and satisfy

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for all } x \in [0, 1], \text{ and}$$

$$|f'_n(x)| \le 1, \quad |f''_n(x)| \le 1 \quad \text{for all } x \in [0, 1],\ n \ge 1.$$

Prove that $f(x)$ is continuously differentiable on $[0, 1]$.

9. (a) Define "f is Riemann integrable on $[0, 1]$".
(b) Prove that every continuous function on $[0, 1]$ is Riemann integrable.

10. Suppose the functions $f_n(x)$ on R satisfy:
(i) $0 \le f_n(x) \le 1$ for all $x \in \mathrm{R}$ and $n \ge 1$.
(ii) $f_n(x)$ is increasing in x for every $n \ge 1$.
(iii) $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in \mathrm{R}$, where f is continuous on R.
(iv) $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 1$.
Show that $f_n(x) \to f(x)$ uniformly on R.

11. (a) Consider the equations

$$u^3 + xv - y = 0, \quad v^3 + yu - x = 0.$$

Can these equations be solved uniquely for u, v in terms of x, y in a neighborhood of $x = 0, y = 1, u = 1, v = -1$? Explain your answer.
(b) Give an example in which the conclusion of the implicit function theorem is true but the hypothesis is not.

12. Let c_0 be the normed space of real sequences $x = (x_1, x_2, ...)$ such that $\lim_{k\to0} x_k = 0$ with the supremum norm $||x|| = \sup_k |x_k|$.
(a) Show that c_0 is complete.
(b) Is the unit ball $\{x \in c_0 : ||x|| \le 1\}$ compact? Prove your answer.
(c) Is the set $\{x \in c_0 : \sum_k k|x_k| \le 1\}$ compact? Prove your answer.