Basic Exam, March 2008

1. Let $g \in C([a,b])$, with $a \leq g(x) \leq b$ for all $x \in [a,b]$. Prove the following:

(i) g has at least one fixed point p in the interval $[a,b]$.

(ii) If there is a value $\gamma < 1$ such that

$$|g(x) - g(y)| \leq \gamma |x - y|$$

for all $x, y \in [a,b]$, then the fixed point p is unique, and the iteration

$$x_{n+1} = g(x_n)$$

converges to p for any initial guess $x_0 \in [a,b]$.

2. Let $\{f_n(x)\}$ be a sequence of continuous functions on the unit interval $[0,1]$ such that $f_n(x) \geq 0$ for all n and x and such that for all $x \in [0,1]$

$$\lim_{n\to\infty} f_n(x) = 0.$$

Prove or give a counterexample to the assertion:

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0.$$

3. Assuming that $f \in C^4[a,b]$ is real, derive a formula for the error of approximation $E(h)$ when the second derivative is replaced by the finite-difference formula

$$f''(x) \sim \frac{f(x+h) - 2f(x) + f(x-h)}{h^2},$$

and h is the mesh size. (Assume that x, $x+h$, $x-h \in (a,b)$).

4. Let X be a compact subset of $\mathbb{R}^N$ and let $\{f_n(x)\}$ be a sequence of continuous real functions on X such that

$0 \leq f_{n+1}(x) \leq f_n(x)$

and

$\lim f_n(x) = 0$ for all $x \in X$.

Prove Dini's Theorem that $f_n(x)$ converges to 0 *uniformly* on X.

5. (a) Let $F(x, y)$ be a continuous function on the plane such that for every square S having its sides parallel to the axes,

$$\int\int_S F(x, y)dxdy = 0.$$

Prove $F(x, y) = 0$ for all (x, y).

(b) Assume $f(x, y)$, $\frac{\partial f(x,y)}{\partial x}$, $\frac{\partial f(x,y)}{\partial y}$, $\frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right)$ and $\frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right)$ are all continuous in the plane. Use part (a) to prove that

$$\frac{\partial}{\partial y}\left(\frac{\partial f(x, y)}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f(x, y)}{\partial y}\right).$$

Hint: You may assume the double integral in (a) equals the iterated integral $\int(\int F(x, y)dx)dy$ and equals the iterated integral $\int(\int F(x, y)dy)dx$.

6. Let Y be a complete *countable* metric space. Prove there is $y \in Y$ such that $\{y\}$ is open.

7. Let $a(x)$ be a function on $\mathbb{R}$ such that

(i) $a(x) \geq 0$ for all x, and

(ii) There exists $M < \infty$ such that for all *finite* $F \subset \mathbb{R}$,

$$\sum_F a(x) \leq M.$$

Prove $\{x : f(x) > 0\}$ is countable.

8. Assume V is an n-dimensional vector space over the rationals $\mathbb{Q}$, and T is a $\mathbb{Q}$-linear tranformation $T : V \to V$ such that $T^2 = T$. Prove that every vector $v \in V$ can be written uniquely as $v = v_1 + v_2$ such that $T(v_1) = v_1$ and $T(v_2) = 0$.

9. Let V be a vector space over $\mathbb{R}$.

(a) Prove that if V is odd dimension, and if T is an $\mathbb{R}$-linear transformation $T : V \to V$ of V, then T has a non-zero eigenvector $v \in V$.

(b) Show that for every even positive integer n, there is a a vector space V over $\mathbb{R}$ of dimension n, and an $\mathbb{R}$-linear transformation $T : V \to V$ of V, such that there is no non-zero $v \in V$ satisfying $T(v) = \lambda v$ for some $\lambda \in \mathbb{R}$.

10. Suppose A is an $n \times n$ complex matrix such that A has n distinct eigenvalues. Prove that if B is an $n \times n$ complex matrix such that $AB = BA$, then B is diagonalizable.

11. Assume A is an $n \times n$ complex matrix such that for some positive integer m the power $A^m = I_n$, where I_n is the $n \times n$ identity matrix. Prove that A is diagonalizable.

12. Let A be an $n \times n$ real symmetric ($a_{i,j} = a_{j,i}$) matrix, and let $S = \{x \in \mathbb{R}^n : \sum x_j^2 = 1\}$ be the unit sphere of $\mathbb{R}^n$. Let $x \in S$ be such that

$$(Ax, x) = \sup_S (Ay, y)$$

where $(z, y) = \sum z_j y_j$ is the usual inner product on $\mathbb{R}^n$. (By compactness such x exists.)

(a) Prove that $(x, y) = 0 \Longrightarrow (Ax, y) = 0$. Hint: Expand

$$(A(x + \epsilon y), x + \epsilon y).$$

(b) Use (a) to prove x is an eigenvector for A.

(c) Use induction to prove $\mathbb{R}^n$ has an orthonormal basis of eigenvectors for A.

Note: If you use part (c) to prove part (a) or part (b), then your solution should include a proof of part (c) that does not use part (a) or part (b).