Basic Examination, Spring 2009

2–6pm, Saturday, March 28, 2009

Instructions: Work any 10 problems. All problems are worth ten points; parts of a problem do not carry equal weight. You must tell us which 10 problems you want us to grade.

The grading will emphasize your attention to detail.

The problems are listed in no particular order.

Problem 1. Set $a_1 = 0$ and define a sequence $\{a_n\}$ via the recurrence

$$a_{n+1} = \sqrt{6 + a_n} \qquad \text{for all } n \geq 1.$$

Show that this sequence converges and determine the limiting value.

Problem 2. Compute the norm of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{bmatrix}$$

That is, determine the maximum value of the length of Ax over all unit vectors x.

Problem 3. We wish to find a quadratic polynomial P obeying

$$P(0) = \alpha, \quad P'(0) = \beta, \quad P(1) = \gamma, \quad \text{and} \quad P'(1) = \delta$$

where $'$ denotes differentiation.

(a) Find a minimal system of linear constraints on $(\alpha, \beta, \gamma, \delta)$ such that this is possible.

(b) When the constraints are met, what is P? Is it unique? Explain your answer.

Problem 4. Let (X, d) be an arbitrary metric space.

(a) Give a definition of *compactness* of X involving open covers.

(b) Define *completeness* of X.

(c) Define *connectedness* of X.

(d) Is the set of rational numbers $\mathbb{Q}$ (with the usual metric) connected? Justify your answer.

(e) Suppose X is complete. Show that X is compact in the sense of part (a) if and only if for every $r > 0$, X can be covered by finitely many balls of radius r.

Problem 5. Compute e^{At} when

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

Recall that e^{At} is defined by the property that a smooth vector function $x(t)$ obeys

$$\frac{dx}{dt}(t) = Ax(t) \quad \Longleftrightarrow \quad x(t) = e^{At}x(0)$$

Problem 6. Show that a continuous function $f : [0, 1) \to \mathbb{R}$ is *uniformly* continuous if and only if there is a continuous function $g : [0, 1] \to \mathbb{R}$ that obeys $f(x) = g(x)$ for all $x \in [0, 1)$.

Problem 7. (a) Define what it means for $f : \mathbb{R}^n \to \mathbb{R}^m$ to be differentiable at a point $a \in \mathbb{R}^n$.
(b) Using this definition, formulate and prove an appropriate form of the chain rule, that is, a theorem describing the derivative of $g(f(x))$ at $x = a$.

Problem 8. Let $M_{n\times n}(\mathbb{R})$ denote the vector space of $n \times n$ matrices with real entries.
(a) Show that

$$\langle A, B\rangle = \mathrm{tr}(AB^T)$$

defines an inner product on $M_{n\times n}(\mathbb{R})$. More precisely, show that it obeys the axioms of an inner product. Note: tr denotes the *trace* of a matrix and ^{T} denotes the *transpose.*
(b) Given $C \in M_{n\times n}(\mathbb{R})$, we define a linear transformation

$$\Phi_C : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R}) \quad \text{by} \quad \Phi_C(A) = CA - AC$$

Compute the adjoint of Φ_C. Check that when C is symmetric, then Φ_C is self-adjoint.
(c) Show that whatever the choice of C, the map Φ_C is not onto (i.e. is not surjective).

Problem 9. Let us say that a real symmetric $n \times n$ matrix A is a *reflection* if $A^2 = \mathrm{Id}$ and

$$\mathrm{rank}(A - \mathrm{Id}) = 1,$$

where Id denotes the identity matrix. Given *distinct* unit vectors $x, y \in \mathbb{R}^n$ show that there is a reflection with $Ax = y$ and $Ay = x$. Moreover, show that the reflection A with these properties is unique.

Problem 10. (a) Rigorously justify the following:

$$\int_0^1 \frac{dx}{1+x^2} = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(-1)^n}{2n+1}$$

(b) Deduce the value of $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$.

Problem 11. (a) Explain the following (overly informal) statement:

> Every matrix can be brought to Jordan normal form; moreover the normal form is essentially unique.

No proofs are required; however, all statements must be clear and precise. All required hypotheses must be included. The meaning of the phrases ‘brought to’, ’Jordan normal form’, and ‘essentially unique’ must be defined explicitly.
(b) Define the *minimal polynomial* of a matrix. How may it be determined for a matrix in Jordan normal form?

Problem 12. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ and $\rho : \mathbb{R}^3 \to \mathbb{R}$ be smooth functions. Show that

$$\mathrm{div}(F) = \rho$$

for all points $(x, y) \in \mathbb{R}^3$ if and only if

$$\iint_{\partial\Omega} F \cdot dS = \iiint_{\Omega} \rho \, dx\, dy\, dz$$

for all balls Ω (with all radii $r > 0$ and all possible centers).
[You may use the various standard theorems of vector calculus without proof.]