Basic Exam, Fall 2012

Instructions: Write your UCLA student number on each page of your solutions. Do not write your name. Work 10 of the 12 problems, and indicate here which 10 problems you want to have graded:

(1), (2), (3), (4), (5), (6), (7), (8), (9), (10), (11), (12).

Each problem is worth 10 points, but different parts of a problem may have different values.

Problem 1. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of real numbers with bounded partial sums i.e., there is $M < \infty$ such that for all N, $|\sum_{n=1}^{N} b_n| \leq M$, and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers decreasing to 0. Prove the series $\sum a_n b_n$ converges.

Problem 2. Let $f(x)$ be a bounded real-valued function on the closed interval $[0, 1]$.

(a) Give a (correct) definition of the Riemann integral $\int_0^1 f(x)dx$ that includes a necessary and sufficient condition for the integral to exist.

(b) Use your answer to (a) to prove that $\int_0^1 f(x)dx$ exists if f is non-decreasing (i.e. $f(x) \leq f(y)$ whenever $x \leq y$).

Problem 3. Let $\{f_n(x)\}$ be a sequence of non-negative continuous functions on a compact metric space X. Assume $f_n(x) \geq f_{n+1}(x)$ for all n and x, so that $\lim_{n\to\infty} f_n(x) = f(x)$ exists for every $x \in X$. Prove f is continuous if and only if f_n converges to f uniformly on X.

Problem 4. A subset K if a metric space (X, d) is called *nowhere dense* if K has empty interior, (i. e., if $U \subset K$, U open in X imply $U = \emptyset$.) Prove the Baire theorem that if (X, d) is a complete metric space, then X is not a countable union of closed nowhere dense sets. Hint: Assume $X = \bigcup_n K_n$ where each K_n is closed and nowhere dense. Show there is $x_1 \in X$ and $0 < \delta_1 < 1/2$ such that $B_1 = B(x_1, \delta_1) = \{y \in X : d(y, x) < \delta_1\}$ satisfies $B_1 \cap K_1 = \emptyset$ and there is $x_2 \in X$ and $0 < \delta_2 < \frac{\delta_1}{2}$ such that $B_2 = B(x_2, \delta_2)$ satisfies $B_2 \subset B_1$ and $B_2 \cap K_2 = \emptyset$. Then continue by induction to find a sequence $\{x_n\}$ in X that converges to $x \in X \setminus \bigcup_{n=1}^{\infty} K_n$.

Problem 5. A subset E of a metric space X is a G_δ set if $E = \bigcap_{n=1}^{\infty} G_n$ where each G_n is open in X. Use Problem 4 to prove that the set of rational numbers is *not* a G_δ subset of the set of real numbers.

Problem 6. (a) Let $F(x,y)$ be a continuous function on the plane such that for every square S having its sides parallel to the axes, $\int\int_S F(x,y)dxdy = 0$. Prove $F(x,y) = 0$ for all (x,y).

(b) Assume

$$f(x,y),\ \frac{\partial f(x,y)}{\partial x},\ \frac{\partial f(x,y)}{\partial y},$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right) \text{ and } \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right)$$

are all continuous in the plane. Use part (a) to prove that

$$\frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right).$$

Hint: You may assume the double integral in (a) equals the iterated integral $\int(\int F(x,y)dx)dy$ and equals the iterated integral $\int(\int F(x,y)dy)dx$.

Problem 7. Let A be an invertible n by n matrix with entries in $\mathbb{C}$. Suppose that the set of powers A^n of A, for $n \in \mathbb{Z}$, is bounded. Show that A is diagonalizable.

Problem 8. Let H be an n by n Hermitian matrix with non-zero determinant. Use H to define an Hermitian form $[,]$ by the formula: for x, y in $\mathbb{C}^n$ (column vectors!), $[x,y] = {}^t\overline{x}Hy$, where, as usual, the bar over x denotes complex conjugation and the t denotes transpose. Let W be a complex subspace of $\mathbb{C}^n$ such that $[w_1, w_2] = 0$ for all, w_1 and w_2 in W. Show that $dimW \leq n/2$. Give also for each n an example of an H for which $dimW = n/2$ if n is even, or $dimW = (n-1)/2$ if n is odd.

Problem 9. Let A be an m by n real matrix with $m \geq n$. Let $b \in \mathbb{R}^m$. Let M be the set of vectors $x \in \mathbb{R}^n$ which minimize $|Ax - b|$. Show that $M = x_0 + N$ where N is the kernel of A, and x_0 is any element of M.

Problem 10. Let A be a linear operator on a four dimensional complex vector space that satisfies the polynomial equation $P(A) = A^4 + 2A^3 - 2A - I = 0$, where I is the identity operator. Let $B = A + I$ and suppose $dim(range(B)) = 2$. Finally, suppose that $|Tr(A)| = 2$. Give a Jordan canonical form of A.

Problem 11. Show that an n by n matrix A can be factored as $A = LU$, where L is lower triangular, and U is upper triangular, provided each determinant $detA_j$, for $j = 1, ..., n-1$, is non-zero, where A_j is the submatrix of A consisting of the first j rows and the first j columns.

Problem 12. Let M be an n by m matrix. Prove that the row rank of M equals the column rank of M. Also, however you prove it, interpret this result as an equality of the dimensions of two vector spaces naturally attached to the map defined by M.