Basic Exam Spring 2014

IMPORTANT. Write your university identification number on the upper right corner of each sheet of paper you use. **Do not write your name anywhere on the exam**

Test Instructions: Work 10 problems, including at least 4 from problems 1 - 6 and at least 4 from problems 7 - 12. Clearly indicate which 10 problems you have attempted. Usually a passing score requires very good work (at least 8 out of 10 points) on at least 3 problems from 1 - 6 and at least 3 problems from 7 -12.

Problem Scores (NG=not graded)

1. Problem 1 __________
2. Problem 2 __________
3. Problem 3 __________
4. Problem 4 __________
5. Problem 5 __________
6. Problem 6 __________
7. Problem 7 __________
8. Problem 8 __________
9. Problem 9 __________
10. Problem 10 __________
11. Problem 11 __________
12. Problem 12 __________

Total ________

Notations:
Let $M_{m,n}(\mathbb{F})$ be the set of all m by n matrices with entries from the field $\mathbb{F}$. Let $\mathrm{Hom}(U,V)$ be the set of all linear maps from the vector space U to the the vector space V.

Problem 1 (a) Find a real matrix A whose minimal polynomial is equal to

$$t^4+1.$$

(b) Show that the usual real linear map determined by $v \mapsto Av$ has no non-trivial invariant subspace.

Problem 2 Suppose that $S, T \in \mathrm{Hom}(V,V)$ where V is a finite dimensional vector space over $\mathbb{R}$. Let $(\mathrm{im}\ S)$ be the image of S and $(\ker S)$ be the kernel of S. Show that

$$\dim(\mathrm{im}\ S) + \dim(\mathrm{im}\ T) \leq \dim(\mathrm{im}\ S \circ T) + \dim V.$$

Problem 3 Suppose that $A, B \in M_{n,n}(\mathbb{C})$ satisfy $AB - BA = A$. Show that A is not invertible.

Problem 4 Suppose that $A, B \in M_{n,n}(\mathbb{C})$. Show that the characteristic polynomials of AB and BA are equal. *Hint:* One approach is to first show that it holds when B is invertible.

Problem 5 Suppose that V is a finite dimensional real inner product space with inner product $\langle \cdot, \cdot \rangle$, that $L \in \mathrm{Hom}(V,V)$ and that $b \in V$ is fixed. Suppose that $u, v \in V$ both minimize $D(x) = ||L(x) - b||$. Show that $u - v \in \ker L$.

Problem 6 Show that if $A \in M_{n,n}(\mathbb{C})$ is normal then $A^* = P(A)$ for some polynomial $P(x)$ with complex coefficients. Here A^* is the conjugate transpose of A.

Problem 7 Find a doubly infinite sequence $\{a_{n,m}, n, m \in \mathbb{Z}\}$ such that for all m, $\sum_n a_{n,m} = 0$ and for all n, $\sum_m a_{n,m} = 0$, with all these series converging absolutely, but such that $\sum_n \sum_m |a_{n,m}| = \infty$.

Problem 8 (a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, & t \leq 0 \\ e^{-\frac{1}{t}}, & t > 0. \end{cases}$$

Prove that f is infinitely differentiable.

(b) In Euclidean space $\mathbb{R}^n$ for $n \geq 1$ find a function $\varphi(x) \in C^\infty$ such that $\varphi(x) \geq 0$ for all x, $\varphi(x) = 0$ if $|x| > 1$, and $\int_{R^n} \varphi(x)dx = 1$.

Problem 9 Find a function that minimizes $\int_0^1 |f'(x)|^2 dx$ among all $f \in C^1(\mathbb{R})$ such that $f(0) = 0, f(1) = 1$. Is the minimizing C^1 function unique on $[0, 1]$?

Problem 10 Let $\mathcal{F}$ be a set of continuous real-valued functions on $[0, 1]$. Assume that

(i) $\mathcal{F}$ is uniformly bounded, i.e. there is $M < \infty$ such that $|f(x)| \leq M$ for all $f \in \mathcal{F}$ and all $x \in [0, 1]$, and

(ii) $\mathcal{F}$ is equicontinuous: for every $\epsilon > 0$ there is $\delta > 0$ such that for all $f \in \mathcal{F}$ and $x, y \in [0, 1]$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

Prove that every sequence from $\mathcal{F}$ has a subsequence that converges uniformly on $[0, 1]$.

Problem 11 Let $\mathcal{F}$ be a set of continuous real-valued functions on $[0, 1]$. Assume that every sequence from $\mathcal{F}$ has a subsequence that converges uniformly on $[0, 1]$. Prove both (i) and (ii) below hold:

(i) $\mathcal{F}$ is uniformly bounded, i.e. there is $M < \infty$ such that $|f(x)| \leq M$ for all $f \in \mathcal{F}$ and all $x \in [0, 1]$, and

(ii) $\mathcal{F}$ is equicontinuous: for every $\epsilon > 0$ there is $\delta > 0$ such that for all $f \in \mathcal{F}$ $x, y \in [0, 1]$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

Problem 12 Assume $[0,1] = \bigcup_{n=1}^{\infty} I_n$ where $I_n = [a_n, b_n] \neq \emptyset$ and

$$I_n \cap I_m = \emptyset$$

whenever $n \neq m$.

(a) Let $E = \{a_n : n \geq 1\} \cup \{b_n : n \geq 1\}$ be the set of endpoints of the intervals above. Prove E is closed.

(b) Prove no such family of intervals $\{I_n\}$ can exist.