Basic Qualifying Exam, 9am–1pm, September 12, 2016

Attempt at most **FIVE** problems numbered 1–6 **AND** at most **FIVE** problems numbered 7–12.

Problem 1. Suppose A, B are invertible $n \times n$ matrices with complex entries, such that $ABA^{-1} = B^3$. Show that all the eigenvalues of B are roots of unity.

Problem 2. Suppose

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Find the smallest possible constant $C > 0$ so that

$$\|e^A x\| \leq C\|x\|$$

for all $x \in \mathbb{R}^2$. Here $\|z\|$ denotes the usual (Pythagorean/Euclidean) length of the vector $z \in \mathbb{R}^2$.

Problem 3. Let

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Find necessary and sufficient conditions on a 3×3 matrix A with real entries so that the equation

$$A = R^T B R$$

admits a solution R of full rank. Here R must be a 2×3 matrix with real entries. You must prove that your conditions are necessary and sufficient.

Problem 4. Let B be an 7×7 matrix and A_n a sequence of matrices of the same size. Suppose that

$$A_n \to B \quad \text{as} \quad n \to \infty,$$

in the sense that the individual matrix entries converge. Show that

$$\operatorname{rank}(B) \leq \liminf_{n\to\infty} \operatorname{rank}(A_n).$$

Problem 5. Let $\alpha_1, \cdots, \alpha_n$ be complex numbers, and $V = \{\sum_{i=1}^n a_i\alpha_i | a_i \in \mathbb{Q}\}$ be the vector subspace of $\mathbb{C}$ spanned by them over the field of rational numbers, $\mathbb{Q}$. Let β be a complex number such that $\beta V \subset V$ where $\beta V = \{\beta v | v \in V\}$. Show that β is the root of a degree n polynomial with rational coefficients.

Problem 6. Let A be an invertible $n \times n$ matrix whose entries belong to $\mathbb{Q}$, the set of rational numbers. Suppose that $Av = 3v$ for a vector $v \in \mathbb{R}^n$.
(i) Give an example in which no non-zero scalar multiple λv for $\lambda \in \mathbb{R}$ is in $\mathbb{Q}^n$.
(ii) Show that there is always a non-zero $w \in \mathbb{Q}^n$ such that $Aw = 3w$.

Problem 7. Determine the volume of the region in $\mathbb{R}^3$ determined by the following inequalities:

$$x^2 + y^2 + z^2 \leq 4, \qquad x^2 - 2x + y^2 + z^2 \geq 0, \quad \text{and} \quad z \geq x.$$

Problem 8. (i) For each $n \in \mathbb{N}$ let $f_n : \mathbb{N} \to \mathbb{R}$ be a function with $|f_n(m)| \leq 1$ for all $m, n \in \mathbb{N}$. Prove that there is an infinite subsequence of distinct positive integers n_i, such that for each $m \in \mathbb{N}$, $f_{n_i}(m)$ converges.
(ii) For n_i as in (i), assume that in addition $\lim_{m\to\infty} \lim_{i\to\infty} f_{n_i}(m)$ exists and equals 0. Prove or disprove: The same holds for the reverse double limit $\lim_{i\to\infty} \lim_{m\to\infty} f_{n_i}(m)$.

Problem 9. Let $f_n : [0, 1] \to \mathbb{C}$ be a sequence of continuous functions. Suppose f_n converge pointwise. Show that the sequence converges uniformly *if and only if* the collection of functions $\{f_n\}$ is equicontinuous.

Problem 10. Find the unique point $(x, y) \in \mathbb{R}^2$ on the curve

$$y^4 + x^4 = 2$$

that is closest to the line

$$y = x - 100.$$

Note: Formal calculations alone do not constitute a solution. You must justify rigorously that there is a point that is closest, that it is unique, and that it is the specific point that you claim that it is.

Problem 11. We define a metric space (X, dist) as follows:

$$X := \{f : [0, 1] \to [0, 1] \,\big|\, f \text{ is continuous and } f(1) = 0.\}$$
$$\mathrm{dist}\big(f, g\big) = \inf\big\{r \in [0, 1] \,\big|\, f(t) = g(t) \text{ for all } r \leq t \leq 1.\big\}$$

Prove any **TWO** of the following statements about (X, dist):

(a) It is not compact. (b) It is not connected.
(c) It is not separable. (d) It is not complete.

Problem 12. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *convex* if and only if

$$f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in \mathbb{R}$ and all $t \in [0, 1]$. Prove the following:
Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Then f is convex if and only if

$$f(y) \geq f(x) + (y - x)f'(x) \quad \text{for all } x, y \in \mathbb{R}.$$