BASIC EXAM: SPRING 2017

Test instructions:

Write your UCLA ID number on the upper right corner of each sheet of paper you use. Do not write your name anywhere on the exam.

Work out 10 problems, including at least 4 of the first 6 problems and at least 4 of the last 6 problems. Clearly indicate which 10 problems you want us to grade.

1	2	3	4
5	6	7	8
9	10	11	12

Problem 1. Let M be an $n \times n$ real matrix and with transpose M^T. Prove that M and $M \cdot M^T$ have the same range (image).

Problem 2. Let $a, b, c, d \in \mathbb{R}$ and

$$M = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & 0 & 0 \\ 0 & c & 3 & -2 \\ 0 & d & 2 & -1 \end{pmatrix}.$$

(i) Determine conditions on a, b, c, d so that there is only one Jordan block for each eigenvalue of M in the Jordan form of M.
(ii) Find the Jordan form of M when $a = c = d = 2$ and $b = -2$.

Problem 3. Let M be an $n \times n$ real matrix. Suppose M is orthogonal and symmetric
(i) Prove that if M positive definite then M is the identity.
(ii) Does the answer change if M is only positive semidefinite?

Problem 4. Let us define $F(x)$ as the following determinant:

$$F(x) = \det \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & x & x^2 & x^3 & x^4 \\ x & 1 & x & x^2 & x^3 \\ x^2 & x & 1 & x & x^2 \\ x^3 & x^2 & x & 1 & x \end{bmatrix}$$

Compute $\frac{dF}{dx}(0)$.

Problem 5. Suppose that V is a finite-dimensional vector space over the field $\mathbb{C}$, and $T : V \to V$ is a linear transformation. Further suppose that $F(X) \in \mathbb{C}[X]$ is a polynomial. Show that the linear transformation $F(T)$ is invertible if and only if $F(X)$ and the minimal polynomial of T have no common factors.

Problem 6. (i) Let V denote the vector space of real $n \times n$ matrices. Prove that the form $\langle A, B \rangle := \text{trace}(A^T B)$, where A^T is the transpose of A, defines an inner product on V. Namely, prove that the form is bilinear, symmetric, and positive definite.
(ii) For $n = 2$, find an orthonormal basis in V.

Problems 7–12 on next page

Problem 7. Prove the *existence* and *uniqueness* of a non-negative continuous function $f : [0,1] \to [0,1]$ satisfying

$$f(x) = 1 - \left[\int_0^x tf(t)\,dt\right]^2$$

Problem 8. Show that there is a constant C so that

$$\left|\frac{f(0)+f(1)}{2} - \int_0^1 f(x)\,dx\right| \le C \int_0^1 |f''(x)|\,dx$$

for every C^2 function $f : \mathbb{R} \to \mathbb{R}$.

Problem 9. Let (X, dist) be a bounded metric space and let $C(X)$ denote the space of bounded continuous real-valued functions on X endowed with the supremum norm. Suppose $C(X)$ is separable.
(i) Show that for every $\varepsilon > 0$, there is a countable set $Z_\varepsilon \subseteq X$ so that

$$\forall x \in X, \quad \exists z \in Z_\varepsilon, \quad \text{such that} \quad \forall y \in X, \quad |\,\mathrm{dist}(x,y) - \mathrm{dist}(z,y)| < \varepsilon.$$

(ii) Deduce that X is separable.

Problem 10. Let $K \subset \mathbb{R}^n$ be compact. Suppose that for every $\varepsilon > 0$ and every pair $a, b \in K$, there is an integer $n \ge 1$ and a sequence of points $x_0, \dots, x_n \in K$ so that

$$x_0 = a, \quad x_n = b, \quad \text{and} \quad \|x_k - x_{k-1}\| < \epsilon \text{ for every } 1 \le k \le n.$$

(i) Show that K is connected.
(ii) Show by example that K may not be path connected.

Problem 11. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of continuous functions satisfying

$$|f_n(x)| \le 1 + \frac{n}{1+n^2x^2}$$

and define

$$F_n(x) = \int_0^x f_n(t)\,dt.$$

Show that there is a subsequence $n_k \to \infty$ so that $F_{n_k}(x)$ converges for every point $x \in [0,1]$.

Problem 12. Show that for each $t \in \mathbb{R}$ fixed, the function

$$F(y;t) = y^4 + ty^2 + t^2 y,$$

defined for all $y \in \mathbb{R}$, achieves its global minimum at a *single* point $y_0(t)$.