BASIC EXAM: FALL 2018

Test instructions:

Write your UCLA ID number on the upper right corner of *each* sheet of paper you use. Do not write your name anywhere on the exam.

The final score will be the sum of **FOUR** analysis problems (Problems 1–6) and **FOUR** linear algebra problems (Problems 7–12). *On the front of your paper indicate which 8 problems you wish to have graded.* Please be reminded that to pass the exam you need to show mastery of both subjects.

Please staple your problems in the order they are listed in the exam.

1	2	3	4
5	6	7	8
9	10	11	12

Problem 1. Let $\{a_n\}_{n\geq 1}$ be a sequence of non-negative numbers such that

$$\sum_{n\geq 1} a_n \quad \text{diverges.}$$

Show that

$$\sum_{n\geq 1} \frac{a_n}{2a_n+1} \quad \text{diverges.}$$

Problem 2. Let A be a connected subset of $\mathbb{R}^n$ such that the complement of A is the union of two separated sets B and C, that is

$$\mathbb{R}^n \setminus A = B \cup C \quad \text{with} \quad \overline{B} \cap C = B \cap \overline{C} = \emptyset.$$

Show that $A \cup B$ is a connected subset of $\mathbb{R}^n$.

Problem 3. Let $f : [0,1] \to \mathbb{R}$ and $g : [0,1] \to [0,1]$ be two Riemann integrable functions. Assume that

$$|g(x) - g(y)| \geq \alpha |x-y| \quad \text{for any} \quad x, y \in [0,1]$$

and some fixed $\alpha \in (0,1)$. Show that $f \circ g$ is Riemann integrable.

Problem 4. Let $f : [0,1] \to \mathbb{R}$ be a continuous function on the closed interval $[0,1]$ and differentiable on the open interval $(0,1)$. Assume that $f(0) = 0$ and f' is a decreasing function on $(0,1)$. Show that

$$g(x) = \frac{f(x)}{x}$$

is a decreasing function on $(0,1)$.

Problem 5. Let $B := \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ and let $g : \partial B \to \mathbb{R}$ be a 1-Lipschitz function.

(a) Show that the function $f : B \to \mathbb{R}$ given by

$$f(x) := \inf_{y \in \partial B} \big[g(y) + |x-y|\big]$$

is 1-Lipschitz.

(b) Show that the set $M(g) := \{h : B \to \mathbb{R} \mid h \text{ is 1-Lipschitz and } h\big|_{\partial B} = g\}$ is compact in the space of continuous functions on B endowed with the supremum norm.

Problem 6. For $x \in (0,\infty)$, let

$$F(x) = \int_0^\infty \frac{1 - e^{-tx}}{t^{\frac{3}{2}}}\, dt.$$

Show that $F : (0,\infty) \to (0,\infty)$ is well-defined, bijective, of class C^1 (i.e. differentiable with continuous derivative), and that its inverse is of class C^1.

Problem 7. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with the property that

$$T(T(x)) = T(x) \qquad \text{for all } x \in \mathbb{R}^n.$$

Show that there exists $1 \leq m \leq n$ and a basis of $\mathbb{R}^n$ such in this basis the entries of T satisfy

$$T_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } 1 \leq i \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 8. Let X be an $n \times n$ symmetric (real) matrix and $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$. Define

$$G = (X - z)^{-1}.$$

Show that

$$\sum_{1 \leq j \leq n} |G_{ij}|^2 = \frac{\operatorname{Im} G_{ii}}{\operatorname{Im} z}.$$

Problem 9. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be linearly independent elements of the vector space (over $\mathbb{R}$) of linear mappings from $\mathbb{R}^n$ to $\mathbb{R}$. Show that for any $v \in \mathbb{R}^n$, there exist v_1 and v_2 such that

$$v = v_1 + v_2, \quad f(v) = f(v_1), \quad \text{and} \quad g(v) = g(v_2).$$

Problem 10. Let $A := \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$. Calculate $\lim_{n \to \infty} A^n$.

Problem 11. Let V be the space of all 3×3 real matrices that are skew-symmetric, i.e. $A^t = -A$ (where A^t denotes the transpose of A). Prove that the expression

$$\langle A, B \rangle = \frac{1}{2} \operatorname{Tr}(AB^t)$$

defines an inner product on V. Exhibit an orthonormal basis of V with respect to this inner product; rigorously justify your answer.

Problem 12. Let V be a finite-dimensional vector space. Let $T : V \to V$ be a linear transformation such that $T(W) \subseteq W$ for every subspace W of V with $\dim(W) = \dim(V) - 1$. Prove that T is a scalar multiple of the identity operator.