Basic Exam: Fall 2019

Test instructions:

- Write your UCLA ID number on the upper right corner of *each* page.
- Do not write your name anywhere on the exam.
- The final score will be the sum of:

  FOUR linear algebra problems (Problems 1–6) and

  FOUR analysis problems (Problems 7–12).

  However, to pass the exam you need to show mastery of both subjects.
- On the front of your paper indicate which 8 problems you wish to have graded.
- Please staple your problems in numerical order.

1	2	3	4	5	6
7	8	9	10	11	12

Problem 1. Let A be an invertible $n \times n$ matrix with real entries and let e_1 denote the unit vector with a 1 in the first position and zeros elsewhere. Show that for each $\lambda \in \mathbb{R}$, the linear transformation A_λ defined by

$$A_\lambda x = Ax + \lambda \langle e_1, x \rangle e_1$$

is invertible if and only if $1 + \lambda \langle e_1, A^{-1} e_1 \rangle \neq 0$.

Problem 2. Find a real symmetric matrix A so that

$$A^2 + A = \begin{bmatrix} 3 & 0 & 0 & 3 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 0 & 0 & 3 \end{bmatrix}.$$

Problem 3. Consider the following two quadratic forms on $\mathbb{R}^3$:

$$Q(x, y, z) = \lambda x^2 + 4y^2 + 16z^2 \quad \text{and} \quad R(x, y, z) = 2xy + 2yz.$$

Determine precisely those values of $\lambda \in \mathbb{R}$ such that there is a linear transformation T on $\mathbb{R}^3$ so that $R = Q \circ T$.

Problem 4. Let V be a vector space and let $1 \leq n < \dim(V)$ be an integer. Let $\{V_i\}$ be a collection of n-dimensional subspaces of V with the property that

$$\dim(V_i \cap V_j) = n - 1 \quad \text{for every } i \neq j.$$

Show that *at least one* of the following holds:
(i) All V_i share a common $(n-1)$-dimensional subspace.
(ii) There is an $(n+1)$-dimensional subspace of V containing all V_i.

Problem 5. Show that an $n \times n$ matrix A with real entries obeys

$$\lim_{k \to \infty} \|A^k\|_{\text{op}} = 0$$

if and only if all (possibly complex) eigenvalues are of modulus strictly less than one. Here $\|\cdot\|_{\text{op}}$ denotes the operator norm, which is defined by

$$\|A\|_{\text{op}} = \sup\big\{\|Ax\| : \|x\| = 1\big\},$$

where $\|\cdot\|$ denotes the usual (Pythagorean/Euclidean) norm on vectors.

Problem 7. Let $\mathcal{M}_n$ denote the vector space of $n \times n$ matrices with real entries. Given $B \in \mathcal{M}_n$, we define a linear transformation $L_B : \mathcal{M}_n \to \mathcal{M}_n$ by

$$L_B(A) = B^T A B \qquad \text{(here } ^T \text{ denotes transpose).}$$

(a) Prove that L_B is invertible if and only if B is invertible.
(b) Find $\text{rank}(L_B)$ in the case that B is diagonal.
(c) Express $\text{rank}(L_B)$ in terms of n and $\text{rank}(B)$ for all choices of B.

Problem 7. Show that the equation

$$x = \cos(x)$$

has exactly one solution on the interval $[0, 1]$.

Problem 8. Show that

$$\sup_{0<h\le 1} \sum_{n\in\mathbb{Z}} \frac{h}{1+n^2h^2} < \infty$$

Problem 9. (a) Show that the relation

$$(2 + x + y)e^z = z^2 + e^x + e^y$$

determines z as a smooth function of (x, y), at least in some neighborhood of the origin.
(b) Show that $(0, 0)$ is a critical point of $z(x, y)$ and determine its nature, that is, whether it is a minimum, a maximum, or a saddle point.

Problem 10. A wheel of radius r rolls in a straight line along flat ground. Determine the length of the path followed by a point on the circumference of the wheel as the wheel makes one revolution.

Problem 11. Let X denote the set of non-decreasing functions $f : [0, 1] \to [0, 1]$. By non-decreasing, we mean that $x \le y \implies f(x) \le f(y)$. We endow X with the following metric:

$$d(f, g) = \sup\big\{|f(x) - g(x)| : x \in [0, 1]\big\}.$$

(a) Prove that (X, d) is complete.
(b) Prove that (X, d) is not compact.

Problem 12. Let $\ell^\infty(\mathbb{Z})$ denote the space of bounded functions $x : \mathbb{Z} \to \mathbb{R}$ together with the metric

$$d(x, y) = \sup_{n\in\mathbb{Z}} |x(n) - y(n)|$$

Show that a function $f : \ell^\infty(\mathbb{Z}) \to \mathbb{R}$ is continuous if and only if its restriction to any compact subset of $\ell^\infty(\mathbb{Z})$ is continuous.