BASIC EXAM: SPRING 2019

Test instructions:

Write your UCLA ID number on the upper right corner of *each* sheet of paper you use. Do not write your name anywhere on the exam.

The final score will be the sum of **FOUR** analysis problems (Problems 1–6) and **FOUR** linear algebra problems (Problems 7–12). *On the front of your paper indicate which 8 problems you wish to have graded.* Please be reminded that to pass the exam you need to show mastery of both subjects.

Please staple your problems in the order they are listed in the exam.

1	2	3	4
5	6	7	8
9	10	11	12

Problem 1. Let $C([0,1])$ denote the space of continuous real-valued functions on $[0,1]$ equipped with the distance

$$d(f,g) = \sup_{x\in[0,1]} \big|f(x) - g(x)\big|.$$

Let X be a subset of $C([0,1])$ defined via

$$X = \{f : [0,1] \to \mathbb{R} :\ f(0) = 0 \text{ and } |f(x) - f(y)| \le |x-y|\}.$$

Show that X is connected and complete.

Problem 2. Let $a, b \in \mathbb{R}$ be two real numbers and consider a sequence $(a_n)_{n\in\mathbb{N}}$ defined recursively by

$$a_0 := a, \quad a_1 := b, \quad a_n := \frac{1}{2}(a_{n-1} + a_{n-2}) \quad \text{for } n \ge 2.$$

Prove that $\lim_{n\to\infty} a_n$ exists and compute its value.

Problem 3. Let $a, b \in \mathbb{R}$ with $a \le b$ and let $f\colon [a,b] \to \mathbb{R}$ be a Riemann integrable function. Assume that there is a $\delta > 0$ such that $f(x) \ge \delta$ for all $x \in [a,b]$. Show that the function $\frac{1}{f}$ is Riemann integrable.

Problem 4. Let $a, b \in \mathbb{R}$ with $a < b$ and let $f, g\colon [a,b] \to \mathbb{R}$ be continuous functions that are differentiable on (a,b). Show that there exists some $\xi \in (a,b)$ such that $(f(b) - f(a))g'(\xi) = (g(b) - g(a))f'(\xi)$.

Problem 5. Show that each metric space can be embedded isometrically into a Banach space. Concretely, if (D,d) is a metric space, then there exist a Banach space $(X, |\cdot|)$ and a map $S\colon D \to X$ with $|S(x) - S(x')| = d(x,x')$ for $x, x' \in D$.

Problem 6. For $n \ge 1$ let $f_n\colon \mathbb{R}_+ \to \mathbb{R}$, $f_n(x) := \frac{x}{n^2}e^{-x/n}$. Show that the sequence (f_n) converges to 0 uniformly over $\mathbb{R}_+$, but $\lim_{n\to\infty} \int_0^\infty f_n(x)dx = 1$.

Problem 7. For

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 3 & -5 & 3 \\ 1 & -1 & 2 \end{bmatrix}$$

write A^{-1} as a polynomial in A with real coefficients.

Problem 8. Let V be the vector space of all 2×2 matrices with real entries. Let W be the subspace generated by all matrices of the form

$$AB - BA$$

where $A, B \in V$. What is the dimension of W? Justify your answer.

Problem 9. Let V be the vector space over $\mathbb{C}$ of all complex polynomials of degree at most 10. Let $D : V \to V$ be the differentiation operator, so $Df(x) = f'(x)$. Find all eigenvalues and eigenvectors of the operator e^D on V.

Problem 10. Let A be an $n \times n$ complex diagonalizable matrix and I the $n \times n$ identity matrix. Show that the $2n \times 2n$ matrix

$$\begin{bmatrix} A & I \\ 0 & A \end{bmatrix}$$

is not diagonalizable.

Problem 11. Let A be an $n \times n$ complex matrix. Prove that $\operatorname{rank}(A) = \operatorname{rank}(A^2)$ if and only if $\lim_{\lambda \to 0}(A + \lambda I)^{-1}A$ exists, where I is the identity matrix.

Problem 12. Let $A = (a_{i,j})$ be an $n \times n$ real matrix whose diagonal entries $a_{i,i}$ satisfy $a_{i,i} \geq 1$ for each $i = 1, \ldots n$. Suppose also that

$$\sum_{i \neq j} a_{i,j}^2 < 1.$$

Prove that A^{-1} exists.