BASIC EXAM: FALL 2020

Test instructions:

Write your UCLA ID number on the upper right corner of each sheet of paper you use. **Do not write your name anywhere on the exam!!!**

All answers must be justified. If you wish to use a known theorem, make sure to give a full and precise statement.

Work out 10 problems, including at least 4 of the first 6 problems and at least 4 of the last 6 problems. Clearly indicate which 10 problems you want us to grade.

Important: No books, notes, calculators, computers of other printed or electronic materials can be used on the exam.

1	2	3	4
5	6	7	8
9	10	11	12

Problem 1. Let M be an $n \times n$ matrix with rational entries, such that $M^2 = 2I$.

(a) Prove that n is even.

(b) Give an example of such matrix M for $n = 2$.

Problem 2. Let A be an orthogonal $n \times n$ matrix.

(a) Prove that A^3 is orthogonal.

(b) Prove that $A + \frac{1}{2} I$ is invertible.

Problem 3. Let M be a complex 4×4 matrix such that $M^6 = M^4 = 2M^3 - M^2$. Describe all possible Jordan forms of M.

Problem 4. Let A be a 2×2 real matrix with eigenvalues 2 and -1. Consider the set X of 2×2 real matrices B, such that the 4×4 matrix

$$\begin{pmatrix} A & B \\ 0 & A \end{pmatrix}$$

is diagonalizable (over $\mathbb{C}$). Prove that X is a 2-dimensional subspace of the vector space of all 2×2 real matrices.

Problem 5. Let $K = \mathbb{F}_p$ be a finite field with p elements, where p is prime. Let $V = K^9$ be a vector space, and let $W \subset V$ be a subspace of V, such that $\dim(W) = 5$. Compute the number of subspaces $U \subset V$, such that $\dim(U) = 6$ and $\dim(W \cap U) = 3$.

Problem 6. Let $v_1, \ldots, v_k \in \mathbb{R}^n$ satisfy $\langle v_i, v_j \rangle < 0$ for all $1 \leq i < j \leq k$, where $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product in $\mathbb{R}^n$. Prove that $k \leq n + 1$.

Problem 7. Let S be a subset of $\mathbb{R}^3$. Prove that the following two conditions are equivalent:

(i) For every point $p \in S$, there exists an open subset V of $\mathbb{R}^3$ containing p, an open subset U of $\mathbb{R}^2$, and a continuously differentiable, one-to-one function $r : U \to \mathbb{R}^3$ such that $r(U) = V \cap S$.

(ii) For every point $p \in S$, there exists an open subset V of $\mathbb{R}^3$ containing p and a continuously differentiable function $f : V \to \mathbb{R}$ such that $Df \neq 0$ on $V \cap S$, and $f^{-1}(\{0\}) = V \cap S$.

(These are two equivalent ways of saying that S is a smooth (C_1) surface in $\mathbb{R}^3$.)

Problem 8. Find all positive values of x for which the series

$$\sum_{n=1}^{\infty} \frac{(xn)^n}{n!}$$

converges.

Problem 9. Let $f(x)$ be real and continuous on $[0, 1]$. Show that

$$\lim_{n\to\infty} (n+1) \int_0^1 x^n f(x)\, dx \;=\; f(1).$$

Problem 10. Let $\{f_n\}_{n=1}^{\infty}$ be a uniformly bounded equicontinuous sequence of real valued functions on a compact metric space X with distance function $d(\cdot, \cdot)$. Define the functions $g_n : X \to \mathbb{R}$ for $n = 1, 2, 3, \ldots$ by

$$g_n(x) = \max\{f_1(x), \ldots, f_n(x)\}.$$

Show that the sequence $\{g_n\}_{n\geq 1}$ converges uniformly.

Problem 11. For each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$, and assume that these functions are uniformly bounded. That is, there exists $M \in \mathbb{R}$ such that $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$. Let X be a countable subset of $\mathbb{R}$. Show that the sequence $\{f_n\}$ has a subsequence that converges (pointwise) for all $x \in X$.

Problem 12. Let X be an open convex subset of $\mathbb{R}^n$. (Recall that a set is called convex if for any points $x, y \in X$, the line segment from x to y is contained in X.) Let $f : X \to \mathbb{R}$ be a differentiable function.

(1) Show that for any $x, y \in X$, there is a point z, lying on the line segment from x to y, for which

$$f(y) - f(x) = \nabla f(z) \cdot (y - x)$$

(2) Use part (a) to show that if the partial derivatives of f are bounded, then f is uniformly continuous on X.