Name: ______________________

Student ID Number: ______________________

UCLA MATHEMATICS – BASIC EXAM: SPRING 2020

INSTRUCTIONS: Do any 10 of the following questions. If you attempt more than 10 questions, indicate which ones you would like to be considered for credit (otherwise the first 10 will be taken). Each question counts for 10 points. Little or no credit will be given for answers without adequate justification. You have 4 hours. Good luck.

#	Score	Counts in 10?
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
Total		10

1. Prove or disprove: There exists $n \geq 1$ and two $n \times n$ matrices A and B with complex entries such that $AB - BA$ equals the identity matrix.

2. If $A, B \in GL_2(\mathbb{C})$ are invertible 2×2 complex matrices, such that $ABA^{-1} = B^5$, show that all the eigenvalues of B are roots of unity of order dividing 24.

3. Let $A = (a_{i,j})_{i,j=1}^{n}$ be an $n \times n$ matrix. Consider the polynomial

$$P(x) := \det \begin{pmatrix} a_{1,1} + x & a_{1,2} + x & \dots & a_{1,n} + x \\ a_{2,1} + x & a_{2,2} + x & \dots & a_{2,n} + x \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} + x & a_{n,2} + x & \dots & a_{n,n} + x \end{pmatrix}.$$

Express the coefficients of $P(x)$ in terms of A.

4. (i) Prove that if V is a finite dimensional inner product space over $\mathbb{C}$ with inner product $\langle \cdot, \cdot \rangle$, then there is no invertible linear transformation $T : V \to V$ such that $\langle v, Tv \rangle = 0$ for all $v \in V$.

(ii) Show that there exists a finite dimensional inner product space V over $\mathbb{R}$ and an invertible linear transformation $T : V \to V$ such that $\langle v, Tv \rangle = 0$ for all $v \in V$.

5. Let

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$$

be an $n \times n$ matrix. Find an invertible matrix T such that $B = T^{-1}AT$ is diagonal, and find this matrix B.

6. A sequence of matrices $A_n \in M_{m\times m}(\mathbb{R})$ is said to converge to a matrix B if the entries of A_n converge to entries of B. Given a matrix $T \in M_{m\times m}(\mathbb{R})$ show that T^n converges to 0 (the zero $m \times m$ matrix) if and only if all the roots λ of the characteristic polynomial of T are such that $|\lambda| < 1$.

7. (i). Let $I = [0,2]$. If $f : I \to \mathbb{R}$ is a continuous function such that $\int_I f(x)dx = 36$, prove that there is an $x \in I$ such that $f(x) = 18$.
(ii) Let $g : I \times I \to \mathbb{R}$ be a continuous function such that $\int_{I\times I} g(x,y)dxdy = 36$. Prove that there is $(x,y) \in I \times I$ such that $g(x,y) = 9$.

8. Assume that $f : [a, b] \to \mathbb{R}$ is continuous such that $\int_a^b x^n f(x)dx = 0$ for all integers $n \geq 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

9. We define a metric space (X, dist) as follows:

$$X := \{f : [0,1] \to [0,1] \mid f \text{ is continuous and } f(1) = 0.\}$$

$$\mathrm{dist}\big(f, g\big) = \inf\big\{r \in [0,1] \mid f(t) = g(t) \text{ for all } r \le t \le 1.\big\}$$

Prove any **TWO** of the following statements about (X, dist):

(a) It is not compact.
(b) It is not connected.
(c) It is not separable.
(d) It is not complete.

10. (i). Let X be a complete metric space with respect to a distance function d. We say that a map $f : X \to X$ is a *contraction* if for some $0 \leq \lambda < 1$ and all $x, y \in X$: $d(f(x), f(y)) \leq \lambda d(x, y)$. Prove that if f is a contraction then it has a unique fixed point, i.e., an $x \in X$ such that $f(x) = x$.

(ii) Suppose a map f satisfies $d(f(x), f(y)) \leq \frac{d(x,y)^2}{1+d(x,y)}$. Prove that f has a unique fixed point.

11. Let $\mathbb{N}$ denote the positive integers and $a_n = \frac{(-1)^n}{\sqrt{n}}$, and β any real number. Prove that there is a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = \beta.$$

12. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *convex* if

$$f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in \mathbb{R}$ and all $t \in [0,1]$. Prove the following:
(a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Then f is convex if and only if

$$f(y) \geq f(x) + (y-x)f'(x) \quad \text{for all } x, y \in \mathbb{R}.$$

(b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^2 (i.e., twice continuously differentiable) and $f'' \geq 0$. Then f is convex.