Qualifying Exam
GEOMETRY/TOPOLOGY
Spring 2001

For the PhD level, do four problems from each part; where Problem 1 counts as two problems.

For the MA level, do five problems in all, with at least two problems from each part. Again, Problem 1 counts as two problems.

Part I: Manifold Theory

1. (20 points) Suppose M is a compact connected 3-manifold and ω is a nowhere zero 1-form defined on M. Suppose that the distribution $\ker \omega$ is integrable, and $\ker \omega = T\mathcal{F}$ for a foliation $\mathcal{F}$.

(4 points) (a) Show that $\omega \wedge d\omega = 0$.

(4 points) (b) Use a partition of unity to show that there is a 1-form α such that $d\omega = \alpha \wedge \omega$.

(2 points) (c) Show $d\alpha \wedge \omega = 0$.

(6 points) (d) Suppose that α' is some other 1-form satsifying $d\omega = \alpha' \wedge \omega$. Show that $\alpha' = \alpha + g\omega$ for some function g, and that $\alpha \wedge d\alpha = \alpha' \wedge d\alpha'$.

(4 points) (e) Suppose that ω' is a nowhere zero 1-form and $\ker \omega = \ker \omega'$. If $d\omega' = \gamma \wedge \omega$, show that $\alpha \wedge d\alpha - \gamma \wedge d\gamma$ is exact.

2. (10 points) On the compact connected manifold M, suppose α is a p-form and β is an $(n-p-1)$-form. Suppose ∂M has two components: $\partial_0 M$ and $\partial_1 M$. Let i_0 and i_1 be the inclusions of $\partial_0 M$ and $\partial_1 M$ into M. Given that $i_0^*\alpha = 0$ and $i_1^*\beta = 0$, show that

$$\int_M d\alpha \wedge \beta = (-1)^{p+1} \int_M \alpha \wedge d\beta$$

3. (10 points) Suppose $f\colon S^1 \to R^2$ and $g\colon S^1 \to R^2$ are smooth embeddings. Let

$$M = \{(a, b, \vec{v}) \in S^1 \times S^1 \times R^2 \colon f(a) - g(b) = \vec{v}\}.$$

Show that M is a compact submanifold of $S^1 \times S^1 \times R^2$. Let $\pi\colon M \to R^2$ be the projection $\pi(a, b, \vec{v}) = \vec{v}$. Apply Sard's Theorem to π and deduce that for almost every $\vec{v} \in R^2$, $f(S^1)$ is transverse to $g(S^1) + \vec{v}$.

4. (10 points) Suppose that $f\colon M \to N$ is a C^∞ map, M and N are compact connected n-manifolds, and $rank(df) = n$. Show that f is a covering map.

Part II: Algebraic Topology

5. (10 points) Let X be a polyhedron, A a subpolyhedon, $p: \widetilde{X} \to X$ the universal covering space of X and $\bar{A}$ the path component of $p^{-1}(A)$ containing the equivalence class of the constant path at $x_0 \in A$.

(2 points) (a) Give an example in which $\bar{p}: \bar{A} \to A$ (where $\bar{p}$ is the restriction of p) is not the universal covering space of A.

(8 points) (b) Prove that $\bar{p}: \bar{A} \to A$ is the covering space of the kernel of $i_*: \pi_1(A, x_0) \to \pi_1(X, x_0)$ where i is inclusion.

6. (10 points) Let D^n be the unit ball in R^n, S^{n-1} its boundary, and $\mathbf{0} \in R^n$ the origin.

(5 points) (a) Prove that the inclusion $i: (D^n, S^{n-1}) \to (D^n, D^n - \mathbf{0})$ induces an isomorphism $i_*: H_n(D^n, S^{n-1}) \to H_n(D^n, D^n - \mathbf{0})$.

(5 points) (b) Prove that i is not a homotopy equivalence of pairs, that is, there is no map $g: (D^n, D^n - \mathbf{0}) \to (D^n, S^{n-1})$ such that gi and ig are homotopic, as maps of pairs, to identity maps.

7. (10 points) Given a map $f: X \to X$ of a polyhedron, there is an exact sequence

$$\to H_k(X) \xrightarrow{1-f_*} H_k(X) \longrightarrow H_k(T_f) \longrightarrow H_{k-1}(X) \to$$

where T_f is the mapping torus of f. Use the sequence to calculate the homology of the 3-manifold M obtained from $S^2 \times I$ by identifying $(x, 0)$ to $(-x, 1)$ for all $x \in S^2$.

8. (10 points) Let $A \subseteq X \subseteq Q$ and consider

$$H_k(Q, A) \xrightarrow{i_*} H_k(Q, X) \xrightarrow{\partial} H_{k-1}(X, A)$$

where i is inclusion and $\partial[z] = [\partial_k[z]]$ for $\partial_k: H_k(Q, X) \to H_{k-1}(X)$ that comes from the exact sequence of (Q, X).

(4 points) (a) Prove that ∂ is well-defined.

(6 points) (b) Prove that the image of i_* equals the kernel of ∂.

9. (10 points) Let $J(X, x_0) \subseteq \pi_1(X, x_0)$ be the subgroup of cyclic classes, where a class α is *cyclic* if there is a homotopy $\{h_t: X \to X\}$ with $h_0 = h_1 =$ identity such that $[h_t(x_0)] = \alpha$.

(7 points) (a) Prove that $J(X, x_0)$ is contained in the center of $\pi_1(X, x_0)$.

(3 points) (b) Prove that if X is a topological group, then $J(X, x_0) = \pi_1(X, x_0)$.