GEOMETRY-TOPOLOGY QUALIFYING EXAMINATION
September 25, 2002

1. Suppose $P(x, y, z)$, $Q(x, y, z)$, and $R(x, y, z)$ are C^∞ functions on $\mathbb{R}^3$ which vanish identically if $|x| \geq 5$, $|y| \geq 5$, or $|z| \geq 5$. Prove that the volume integral

$$\int_{-6}^{+6} \int_{-6}^{+6} \int_{-6}^{+6} d(Pdy \wedge dz + Qdx \wedge dz + Rdx \wedge dy) = 0.$$

(Do this directly, not by quoting Stokes' Theorem: this is a special case of the proof of Stokes' Theorem!)

2. Suppose that $V = P(x, y, z)\frac{\partial}{\partial x} + Q(x, y, z)\frac{\partial}{\partial y} + R(x, y, z)\frac{\partial}{\partial z}$ is a C^∞ vector field on $\mathbb{R}^3$ with $V \neq \vec{0}$ at the origin. Find a necessary and sufficient condition for there to exist a C^∞ function $\lambda(x, y, z)$ in some neighborhood of the origin such that λV is the gradient of a C^∞ function on the neighborhood.

3. Let $T_t : \mathbb{R}^3 \to \mathbb{R}^3$ be the right-hand rule rotation around the positive z-axis by t degrees and $S_s : \mathbb{R}^3 \to \mathbb{R}^3$ be the right-hand-rule rotation around the positive x-axis by t degrees.

(a) Find the infinitesimal generators of the flows T_t and S_t, i.e., the vector fields X and Y, respectively, on $\mathbb{R}^3$ whose flows are $\{T_t\}$ and $\{S_t\}$.

(b) Compute the commutator

$$T_{-t} \circ S_{-t} \circ T_t \circ S_t.$$

(c) Compare the result of (b) (lowest order non-identically zero term) with the Lie bracket $[X, Y]$.

4. Take as given that a C^∞ 2-form ω on S^2 is of the form $d\theta$ for some C^∞ 1-form θ if and only if $\int_{X^2} \omega = 0$. Use this to show that every C^∞ 2-form Ω on $\mathbb{R}P^2$ has the form $d\Lambda$ for some C^∞ 1-form Λ. (Do not just quote DeRham's Theorem here.)

5. (a) Suppose $F : S^1 \to \mathbb{R}^3$ is a C^∞ function such that dF is nowhere zero (on S^1). Prove that there is a two-dimensional subspace P of $\mathbb{R}^3$ such that $\pi_P \circ F : S^1 \to \mathbb{R}^3$ has nowhere vanishing differential, where $\pi_P =$ orthogonal projection on P.

(b) Show by example (a picture with explanation is all right) that there is such an F that is also 1 to 1 (injective) but is such that, for all P, $\pi_P \circ F$ fails to be injective.

(c) Show that if $F : S^1 \to \mathbb{R}^4$ is C^∞ and injective then there is a three-dimensional subspace H of $\mathbb{R}^4$ such that $\pi_H \circ F$ is injective, where π_H = orthogonal projection on H.

6. (a) Suppose $F : S^n \to S^n$ is fixed-point free (i.e., for all $p \in S^n$, $p \neq F(p)$). Show that F is homotopic to the antipodal map $p \to -p$, $p \in S^n$.

(b) Use part (a) to show that every vector field on (tangent to) S^{2n}, $n = 1, 2, 3 \ldots$, vanishes somewhere on S^{2n} (i.e., has a zero).

7. (a) Discuss carefully how to obtain the long exact sequence in homology from a short exact sequence of chain complexes. (Include definitions of the maps in the long exact sequence.)

(b) If the short exact sequence is

$$0 \to C_1 \to C_2 \to C_3 \to 0,$$

prove exactness of the long exact sequence at $H_k(C_3)$ [in $\ldots H_k(C_2) \to H_k(C_3) \to H_{k-1}(C_1) \ldots$].

8. (a) Suppose $F : T^2 \to T^2$ (where $T^2 = S^1 \times S^1$) is a continuous function such that $F(p) = p$ for some $p \in T^2$ and

$$F_* : \pi_1(T^2, p) \to \pi_1(T^2, p)$$

is the identity map. Is F necessarily homotopic to the identity map from T^2 to itself?

(b) Is a C^∞ map $F : T^2 \to T^2$ of degree 1 necessarily homotopic to the identity map of T^2 to itself? Explain/prove your answer.

9. (a) Discuss the (a) representation of $\mathbb{C}P^n$ as a cell complex.

(b) Use part (a) to find the homology of $\mathbb{C}P^n$: prove carefully that your calculation is correct.

10. (a) Let X = the space obtained by attaching two discs to S^1, the first disc being attached by $S^1 = \partial D_1 \to S^1$ being the 7 times around (counterclockwise) map, e.g., $z \to z^7$, $|z| = 1$, $z \in C$ and the second being attached by $S^1 = \partial D_2 \to S^1$ being the 5 times around map $z \to z^5$. Find the homology of X.

(b) Can X be made a C^∞ manifold? Why or why not?