M.A. or Ph.D. Qualifying Exam
GEOMETRY-TOPOLOGY
Spring 2002

Instructions:

For a Ph.D. pass do 4 problems from each section to a total of 8 problems. For a M.A. pass do 2 from one and 3 from the other section to a total of 5 problems.

Geometry

1. Let M be a closed (compact, without boundary) manifold. Show that any smooth function

$$f : M \to \mathbb{R}$$

has a critical point.

2. (a) Show that every closed 1-form on S^n, $n > 1$, is exact.

(b) Use this to show that every closed 1-form on $\mathbb{R}P^n$, $n > 1$, is exact.

3. Let M^d be a d-dimensional manifold and $\omega_1, \ldots, \omega_p$ be pointwise linearly independent 1-forms. If $\theta_1, \ldots, \theta_p$ are 1-forms so that

$$\sum_{i=1}^{p} \omega_i \wedge \theta_i = 0,$$

then there exist smooth functions f_{ij} so that

$$\theta_i = \sum_{i=1}^{p} f_{ij}\omega_j, \quad i = 1, \ldots, p.$$

(**Hint:** try $p = 1$)

4. Let M be the set of all straight lines in $\mathbb{R}^2$ (not just those which pass through the origin). Show that M is a smooth manifold and identify it with a well-known manifold.

(**Hint:** Lines not through the origin have a unique closest point to the origin and that point determines the line uniquely. What happens at the origin?)

5. Let $f : M^m \to N^n$ be a smooth bijection so that $Df : T_pM \to T_{f(p)}N$ is injective for all p. Show that f is a diffeomorphism.

Topology

6. (a) Show that if $f : S^n \to S^n$ has no fixed points then $\deg(f) = (-1)^{n+1}$.

   (b) Show that if X has S^{2n} as universal covering space then $\pi_1(X) = \{1\}$ or $\mathbb{Z}_2$.

   (c) Show that if X has S^{2n+1} as universal covering space then X is orientable.

7. (a) Outline the construction of the universal covering of a path connected locally simply connected space X.

   (b) Give an example of a path connected space which does not have a universal covering space.

8. Let X be a finite cell complex constructed inductively by gluing all p-cells onto cells of dimension $< p$. Assume no $p-1$ and $p+1$ cells are used to construct X. Show that

$$H_p(X, \mathbb{Z}) \simeq \mathbb{Z}^{n_p}$$

when n_p is the number of p-cells used in the construction.

9. Let $(M, \partial M)$ be a compact oriented n-manifold with connected boundary ∂M. Show that there is no retract $r : M \to \partial M$, i.e., a map $r : M \to \partial M$ such that $r(x) = x$ if $x \in \partial M$.
(Hint: Prove that $H_{n-1}(\partial M) \to H_{n-1}(M)$ is trivial.)

10. Let $X = T^2 - \{p, q\}$, $p \neq q$ be the twice punctured 2-dimensional torus.

   (a) Compute the homology groups $H_*(X, \mathbb{Z})$.

   (b) Compute the fundamental group of X.