Qualifying Exam
Geometry/Topology
Winter 2002

Instructions:

For the PhD level, do four problems from each part.

For the MA level, do five problems in all, with at least two problems from each part.

Part I: Differentiable Manifolds

1. Let M be a smooth three dimensional manifold and α is a 1-form on M s.t. $\alpha \wedge d\alpha \neq 0$ at every point of M. (10 points)

(i) Let $H = \ker\ \alpha \subseteq TM$. Show that H is a two-dimensional plane field of TM which is not integrable.

Hint: Use the formula $d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y])$, where X, Y are two arbitrary vector fields.

(ii) Show that there exists a unique vector field V s.t.

(a) $\alpha(V) = 1$, (b) $\langle V \rangle \oplus H = TM$, (c) $d\alpha(V, W) = 0$

for any vector field W. Here $\langle V \rangle$ is the line field generated by V.

2. Let M be a closed smooth manifold and X be a vector field on M. Denote the flow generated by X by $\varphi_t : M \to M$, i.e., φ_t is defined by: (10 points)

$$\frac{d\varphi_t}{dt}(x) = X(\varphi_t(x)) \quad \text{for any} \qquad x \in M.$$

Given a function f, prove that:

$$f \circ \varphi_1 - f \circ \varphi_0 = \int_0^1 \varphi_t^*(df)(X)dt.$$

3. Let M_n be the space of $n \times n$ real matrices and M_n^k be the subspace of all matrices of rank k in M_n. (10 points)

(i) Show that M_n^k is a submanifold of M_n.

(ii) Find the dimension of M_n^k.

4. Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ as usual. (10 points)

(a) Show that, for each C^∞ 1-form ω on S^2 with $d\omega = 0$, there is a C^∞ function $f : S^2 \to \mathbb{R}$ such that $df = \omega$.

(b) Show that, for each 2-form Ω on S^2 such that $\Omega = d\theta$ for some 1-form θ,

$$\int_{S^2} \Omega = 0.$$

(c) Is the converse of (b) true, i.e., is it true that if Ω is a 2-form on S^2 with $\int_{S^2} \Omega = 0$ then there is always a 1-form θ on S^2 such that $\Omega = d\theta$? Prove your answer.

5. Let S^2 be as in Problem 4. Consider the 2-form on $\mathbb{R}^3 - \{(0, 0, 0)\}$ (10 points)

$$\sigma = (x^2 + y^2 + z^2)^{-3/2}(x\ dy \wedge dz - y\ dx \wedge dz + z\ dx \wedge dy).$$

(a) Show that σ is closed on $\mathbb{R}^3 - \{(0, 0, 0)\}$.

(b) Show that the 2-form

$$\omega = x\ dy \wedge dz - y\ dx \wedge dz + z\ dx \wedge dy$$

is closed but not exact on S^2.

(c) Find $\int_{S^2} \omega$.

(d) Suppose M is compact, 2-dimensional, oriented embedded submanifold of $\mathbb{R}^3 - \{(0, 0, 0\}$. What are the possible values of $\int_M \sigma$? Prove your answer.

Part II: Algebraic Topology

6. (a) Define: chain complex, chain map, chain homotopy. (10 points)

(b) Prove that if $f_1, f_2 : C \to C'$ and $g_1, g_2 : C' \to C''$ are chain homotopic chain maps then $g_1 \circ g_1,\ g_2 \circ f_2 : C \to C''$ are also chain homotopic.

7. Let $p : \widetilde{X} \to X$ be a covering space and let $f : X \to X$ be a map such that $f(x_0) = x_0$. A map $\widetilde{f} : \widetilde{X} \to \widetilde{X}$ such that $f(\widetilde{x}_0) = \widetilde{x}_0$ for some $\widetilde{x}_0 \in p^{-1}(x_0)$ is a *lift* of f if $p\widetilde{f} = fp$. (10 points)

(a) Prove that f has a *lift* if and only if $f_*(H) \subseteq H$ where $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subseteq \pi_1(X, x_0)$.

(b) Give an example of a space X, a map $f : X \to X$ and a covering space $p = \widetilde{X} \to X$ such that f has *no lifts* to $\widetilde{X}$.

8. The following diagram of groups and homomorphisms is commutative and both horizontal sequences are exact. The symbol "id" denotes the identity. Prove that if $c \in C$ such that $\gamma(c) = 1$ then there exists $b \in B$ such that $\beta(b) = 1$ and $\varphi(b) = c$, and thus that $\varphi(\ker \beta) = \ker \gamma$. (10 points)

$$\begin{array}{ccccccc}
A & \xrightarrow{\alpha} & B & \xrightarrow{\varphi} & C & \xrightarrow{\delta} & D \\
{\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \beta}\downarrow & & {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
A & \xrightarrow[\alpha']{} & B' & \xrightarrow[\varphi']{} & C' & \xrightarrow[\delta']{} & D
\end{array}$$

9. Let (X_1, A_1) and (X_2, A_2) be pairs of finite polyhedra and subpolyhedra. (10 points)

(a) Write the *relative* Mayer-Vetoris sequence for the pair $(X_1 \cup X_2,\ A_1 \cup A_2)$. You do not have to define the homomorphism or prove anything about it.

(b) Use part (a) to prove that if X is a finite polyhedra, S^r is the r-sphere, $p_0 \in S^r$ and $k > r$ then

$$H_k(X \times S^r,\ X \times p_0) \simeq H_{k-r}(X).$$

10. Let $p : E \to B$ be a covering space and $f : X \to B$ a map. Define (10 points)

$$E^* = \{(x, e) \in X \times B : f(x) = p(e)\}.$$

Prove that $q = E^* \to X$ defined by $q(x, e) = x$ is a covering space.