M.A. or Ph.D. Qualifying Exam
GEOMETRY/TOPOLOGY
Fall 2003

Instructions: All problems are worth ten points.

1. Explain carefully how the classical "divergence theorem"

$$\iint_S \vec{V} \cdot \vec{n} \; d(\text{area}) = \iiint_V \text{div}\, \vec{V} \; d(\text{volume})$$

(V a bounded volume in $\mathbb{R}^3, S =$ boundary of V) follows from the Stokes' Theorem for differential forms.

2. Without using deRham's Theorem, prove:

(a) every closed 1-form on S^2 is exact.

(b) a two-form Ω is exact on S^2 if and only if

$$\int_{S^2} \Omega = 0.$$

3. Show that the set of all lines in $\mathbb{R}^2$ has a natural structure as a differentiable manifold. What (already familiar) manifold is it?

4. Show that S^3 is the union of two solid tori ($S^1\times$ 2-disc) with a embedded torus ($S^1 \times S^1$) as their common boundary. (**Hint:** Express $\mathbb{R}^3$ with a solid torus removed as a union of circles and a single straight line and then add a point at infinity.)

5. Suppose M is a compact manifold (with empty boundary).

(a) Prove that, if $f : M \to \mathbb{R}$ is a C^∞ function, then f has at least two critical points.

(b) A C^∞ function on $S^1 \times S^1$ cannot have only two critical points. Prove this (e.g.) by deforming a homotopically nontrivial S' along the gradient flow of $f : M \to \mathbb{R}$.

6. (a) Prove carefully that a group of homeomorphisms of S^{2n}, each of which has no fixed points (unless it is the identity map), contains at most two elements.

(b) Give a counterexample for some S^{2n+1}, $n \geq 1$.

7. Find the homology groups with $\mathbb{Z}$ coefficients, of $\mathbb{R}P^n$, $n = 2, 3, 4 \ldots$ by some systmatic rigorous method.

8. Find the homology and the fundamental group of $S^1 \times S^1$ with two points removed.

9. Prove that if a compact (empty boundary) manifold X has S^{2n+1}, $n \geq 1$, as a covering space, then X is orientable.

10. Suppose M is a compact orientable manifold (empty boundary). Prove that

$$H_n(M,\ \mathbb{Z}) \simeq \mathbb{Z}.$$

(You may assume M is triangulated.)