GEOMETRY/TOPOLOGY QUALIFYING EXAMINATION

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Manifold Theory

1. Let M be a smooth compact manifold of dimension n. Show that there is no immersion of M into $\mathbb{R}^n$.

2. The n-dimensional torus T^n is defined to be $\mathbb{R}^n/\mathbb{Z}^n$, i. e. for any x and y in $\mathbb{R}^n$, $x \sim y$ iff $x - y \in \mathbb{Z}^n$. Let α and β be two functions on $\mathbb{R}^n$ such that (i) $\alpha(x) = \alpha(y)$ and $\beta(x) = \beta(y)$ iff $x - y \in \mathbb{Z}^n$ and (ii) α/β is an irrational constant. Then

$$v = \alpha(x)\frac{\partial}{\partial x^1} + \beta(x)\frac{\partial}{\partial x^2}$$

is a vector field on $\mathbb{R}^n$ descending to T^n, where

$$\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n}\}$$

are coordinate vector fields. Find all functions f on T^n such that $vf = 0$.

3. Let M and N be smooth compact connected manifolds and $f: M \to N$ be a smooth map such that, for any point $m \in M$, $rank(df_m) = \dim(N)$. Show that (i) for any $n \in N$, $f^{-1}(n)$ is a submanifold of M and (ii) for any n_1 and n_2 in N, the submanifolds $f^{-1}(n_1)$ and $f^{-1}(n_2)$ of M are diffeomorphic to each other.

4. Let

$$\tilde{\theta} = \frac{1}{2}\{(x^2dx^1 - x^1dx^2) + (x^4dx^3 - x^3dx^4) + \cdots + (x^{2n}dx^{2n-1} - x^{2n-1}dx^{2n})\}$$

be a 1-form on $\mathbb{R}^{2n}$ and θ be its restriction to the unit sphere

$$S^{2n-1} = \{x = (x^1, \dots, x^{2n}) | (x^1)^2 + \cdots + (x^{2n})^2 = 1\}.$$

The kernel K of θ is a distribution on S^{2n-1}:

$$K = \{v | v \in TS^{2n-1}, \theta(v) = 0\}.$$

Decide whether or not K is integrable.

5. Let $T^{2m} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ be torus of dimension $2n$. Consider the 2-form

$$\omega = dx^1 \wedge dx^{n+1} + dx^2 \wedge dx^{n+2} + \cdots + dx^n \wedge dx^{2n}$$

defined on $\mathbb{R}^{2n}$ descending to T^{2n}.

(i) Show that ω is closed but not exact on T^{2n}.

(ii) Let $i\colon T^n \to T^{2n}$ be the subtorus defined by the equation

$$x^{n+1} = x^{n+2} = \cdots = x^{2n} = 0.$$

What is $i^*\omega$?

(iii) Let $\Sigma = S^2 \setminus \{\cup_{i=1}^m D_i\}$, where D_i, $i = 1, \ldots, m$ are m open disks in S^2 with disjoint closures. Show that

$$\int_\Sigma f_1^*\omega = \int_\Sigma f_2^*\omega$$

if $f_1, f_2\colon (\Sigma, \partial\Sigma) \to (T^{2n}, T^n)$ are homotopic to each other, where $\partial\Sigma$ is the boundary of Σ.

Algebraic Topology

1. Let X be a path connected space and let $x_0, x_1 \in X$. Prove carefully that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

2. (i) Define what is meant by a "chain homotopy" P between chain maps $f_\#, g_\#\colon C \to D$ and prove that chain homotopic chain maps induce the same homomorphism of homology. (ii) Let X and Y be spaces and let $F\colon X \times I \to Y$ be a homotopy between maps f and g. Define a chaim homotopy P between the induced chain maps $f_\#, g_\#\colon C(X) \to C(Y)$ of singular chains. (iii) Verify that P satisfies the definition of a chain homotopy ONLY for the restriction of P to $C_1(X)$.

3. Let X be a locally contractible space and H a subgroup of $\pi_1(X, x_0)$. Describe carefully how to construct a topological space X_H and a map $p\colon X_H \to X$ such that $p_*(\pi_1(X_H, \tilde{x}_0)) = H$ and show that it has the required property of p_*. (Note: Although X_H will be a covering space, you don't have to verify this unless you want to use some general properties of covering spaces.)

4. Prove that the real even-dimensional projective spaces $\mathbb{R}P^{2n}$ have the fixed point property, that is, every for every map $f\colon \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ there is a solution to $f(x) = x$. (Hint: Consider maps on the covering space S^{2n}.)

5. Use the Mayer-Vietoris sequence to calculate the homology of $S^1 \times S^2$. You may assume the homology calculations for $S^1, S^1 \times S^1$ and S^2.