M.A. or Ph.D

Qualifying Exam

Geometry/Topology

Fall 2004

Instruction: All problems are worth ten points.

1. Let M be a connected smooth manifold. Construct the orientation cover M_0.

   a) Show that M_0 is a smooth manifold.

   b) Show that M_0 is a 2:1 covering of M.

   c) Show that M is orientable iff M_0 is the union of two disconnected components.

2. Let ω be a smooth nowhere vanishing 1-form on a smooth connected manifold M.

   a) Show that $\ker \omega$ is a smooth co-dimension 1 distribution on M.

   b) Show that $\ker \omega$ is integrable iff $d\omega$ vanishes on $\ker \omega$.

   c) Find a co-dimension 1 distribution on $\mathbb{R}^3$ that is not integrable.

3. Show that $S^1 \times S^n$ is parallelizable, i.e., one can find $(n+1)$ vector fields that are everywhere linearly independent.
   ($S^k \subset \mathbb{R}^{k+1}$ is the unit sphere)

4. Let $\omega = \frac{-ydx+xdy}{(x^2+y^2)^\alpha}$ and consider $\int_\gamma \omega$, where $\gamma : S^1 \to \mathbb{R}^2 - \{0\}$.

   a) For which α is $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$, whenever γ_1 and γ_2 are smoothly homotopic, i.e., then exists $F : S^1 \times [0,1] \to \mathbb{R}^2 - \{0\}$ such that $\gamma_0(t) = F(t,0)$, $\gamma_1(t) = F(t,1)$?

   b) What are the possible values for $\int_\gamma \omega$ when α is chosen as in part a)?

5. Show that a closed (compact without boundary) n-manifold cannot be immersed in $\mathbb{R}^n$.

6. Let $\mathbb{C}^*$ be the set of all non-zero complex numbers with the induced topology from $\mathbb{C}$. It is a topological group with respect to the usual multiplication. Let f be a continuous homomorphism from C^* to itself.

   (i) Find all possible $f|_{S^1}$, where $S^1 = \{z \mid |z| = 1, z \in \mathbb{C}^*\}$.

   (ii) Classify such $f|_{S^1}$ up to homotopy.

7. Let $X_1 = S^1 \vee_{x_1=x_2} S^2$ be the space obtained from the disjoint union of the circle S^1 and the S^2 by identifying a point $x_1 \in S^1$ with a point $x_2 \in S^2$. Define $X_2 = S^1 \vee_{x_1=x_2} S^1$ similarly.

   (i) Find $\pi_1(X_1)$ and $\pi_1(X_2)$.

   (ii) Find their universal coverings.

8. Let $f : S^2 \to T^2$ be a continuous map from 2-sphere to 2-torus T^2.
   What is the induced map

$$f_* : H_*(S^2) \to H_*(T^2)$$

   on the homology groups?

9. Let X be a topological space, and define $S(X)$ to be the quotient space of $X \times I$ by contracting $X \times \{0\}$ to a point and $X \times \{1\}$ to another point. Here $I = [0, 1]$. What is the relationship between $H_*(S(x))$ and $H_*(x)$?

10. Let K be a finite simplicial complex and K^n be the subcomplex consisting of all simplices in K of dimension less than or equal to n. Denote the underlying topological spaces of K and K^n by $|K|$ and $|K^n|$.

    (i) What is the relative singular homology $H_*(|K^n|, |K^{n-1}|)$?

    (ii) Write down the long exact sequence for the triple $(|K^n|, |K^{n-1}|, |K^{n-2}|)$, i.e. , the long exact sequence relating the singular homology groups $H_*(|K^n|, |K^{n-1}|)$, $H_*(|K^{n-1}|, |K^{n-2}|)$ and $H_*(|K^n|, |K^{n-2}|)$.

    (iii) Use (i) and (ii) to show that singular homology of $|K|$ is same as the simplicial hmology of $|K|$. (Hint: identify the connecting boundary map in (ii)).