GEOMETRY/TOPOLOGY QUALIFYING EXAMINATION

Winter, 2004

Manifold Theory

1. (a) Let $M = SL(2,\mathbb{R}) = \{A \in M_2\mathbb{R}; \det A = 1\}$. Show that M is a submanifold of $M_2(\mathbb{R})$ (the space of two-by-two matrices). Given $A \in M$, regard T_AM as a subspace of $M_2\mathbb{R}$. Consider three vector fields H, X, Y on M defined by

$$H(A) = A \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X(A) = A \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y(A) = A \cdot \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \in T_AM.$$

Find the flows of H, X and Y.

(b) Show that $[H, X] = 2X$.

2. State the general Stokes Theorem, and explain how the classical version

$$\int\int_S (\nabla \times \vec{v}) \cdot \vec{n}\, dA = \int_{\partial S} \vec{v} \cdot d\vec{r}$$

follows. Here S is a compact surface in $\mathbb{R}^3$ with normal vector $\vec{n}$ and boundary ∂S, and $\vec{r}$ is the position vector.

3. Describe diffeomorphisms between $SO(3), \mathbb{R}P^3$ and $UT(S^2)$, the unit tangent bundle of S^2. You need not check that the maps are smooth. ($SO(3)$ is the special orthogonal group and $UT(S^2)$ is the set of tangent vectors of length one.)

4. Let X be the space of symmetric n-by-n real matrices and let X_k be the subspace of matrices of rank k in X. Show that X_k is a submanifold and find its dimension.

5. Suppose that $f: M \to N$ is C^∞, M and N are compact connected n-manifolds, and $rank(df) = n$. Show that f is a covering map.

Algebraic Topology

6. Consider the exact sequence of abelian groups and homomorphisms

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0.$$

Prove that if there is a homomorphism $\gamma: B \to A$ such that $\gamma\alpha: A \to A$ is the identity, then B is isomorphic to $A \oplus C$.

7. Prove that the n-sphere S^n admits a continuous field of nonzero tangent vectors if and only if n is odd.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

8. Let $p: \widetilde{X} \to X$ be the universal covering space of a space X and let $f: X \to X$ be a map.

(a) Prove that there exist *lifts* of f to $\widetilde{X}$, that is, maps $\tilde{f}: \widetilde{X} \to \widetilde{X}$ such that $p\tilde{f} = fp$.

(b) Suppose $\tilde{f}_1, \tilde{f}_2$ are lifts of f and there exist $\tilde{x}_1, \tilde{x}_2 \in \widetilde{X}$ such that $\tilde{f}(\tilde{x}_1) = \tilde{x}_1, \tilde{f}(\tilde{x}_2) = \tilde{x}_2$ and $p(\tilde{x}_1) = p(\tilde{x}_2)$. Prove that there exists a covering transformation $\sigma: \widetilde{X} \to \widetilde{X}$ such that $\tilde{f}_2 = \sigma \tilde{f}_1 \sigma^{-1}$.

9. Let $X_k = S^1 \times D^2 - \{p_1, p_2, \ldots, p_k\}$ be the solid torus (circle cross disc) with $k \geq 1$ points deleted from its interior. Calculate the homology of X_k.

10. Let $\Omega(X)$ denote the *loop space* of a metric space X with metric d. That means $\Omega(X)$ is the set of all maps $a: [0,1] \to X$ such that $a(0) = a(1)$, with the topology given by the metric $d(a,b) = \max_{0 \leq t \leq 1} d(a(t), b(t))$. Suppose $a, b \in \Omega(X)$ such that $a(0) = b(0) = x_0$. Prove that the classes $[a], [b] \in \pi_1(X, x_0)$ are conjugate in $\pi_1(X, x_0)$ if and only if a and b lie in the same path component of $\Omega(X)$.