Geometry/Topology

QUALIFYING EXAM

September 2005

Do all problems

Manifold Theory

1. (a) Define complex projective space , P^n, $n \geq 1$, and prove that it is a compact differentiable manifold.
   (b) Show that P^n is orientable for all n.
   (c) Prove that P^1 is diffeomorphic to S^2.

2. Suppose N is an embedded submanifold of a (C^∞) manifold M. A vector field X on M is *tangent to* N if $X(p) \in T_pN \subset T_pM$ for all $p \in N$.
   (a) Prove that if X and Y are vector fields on M that are each tangent to N, then $[X, Y]$ is also tangent to N.
   (b) Illustrate this principle for two vector fields (your choice) tangent to $S^2 \subset {}^3$ (with $[X, Y] \not\equiv 0$), computing $[X, Y]$ and checking that $[X, Y]$ is tangent to S^2

3. DeRham's Theorem says that (closed p-forms)/(exact p-forms) $\cong H^p(M, \ )$ for a C^∞ manifold M. Verify this in the special case $M = S^2$, where $H^1 = 0$, $H^2 = 1$ (given) by proving, by direct constructions:
   (a) Every closed 1-form ω (i.e., $dw = 0$) is exact (i.e., $\omega = df$, some function f).
   (b) A (closed, necessarily) 2-form Ω on S^2 is $d\Theta$, for some 1-form Θ, if and only if $\int_{S^2} \Omega = 0$.
   (c) There is a 2-form Ω on S^2 such that $\int_{S^2} \Omega \neq 0$.

4. A vector field V on ^{3} is said to be *gradient-like* at a point $(x, y, z) \in {}^3$ if there is a neighborhood U of (x, y, z) and a nowhere-vanishing -valued function λ on U with the property that $\text{curl}(\lambda V) \equiv \vec{0}$ on U. (So λV is the gradient of a function in a neighborhood of (x, y, z).)
   Use the Frobenius Theorem to find a condition under which a nowhere vanishing vector field V on ^{3} is gradient-like at each $(x, y, z) \in {}^3$. Demonstrate that your condition works

5. Prove that if M is a compact C^∞ manifold, then for some positive integer k there is a C^∞ mapping $F : M \to {}^k$ such that $dF|_q$ is injective for all $q \in M$.

Algebraic Topology

6. (a) Write down the Mayer-Vietoris sequence associated to a pair of open sets U and V with $U \cup V =$ a topological space X.
   (b) Describe explicitly how the dimension-lowering map(s) in this long exact sequence arise and prove that this map is well-defined.

7. Let X be a space that has a simply-connected covering space $p : \tilde{X} \to X$.

(a) Prove that X is semi-locally simply connected. (i.e. each $x \in X$ has a neighborhood such that every loop at x that is in the neighborhood is contractible in X.)

(b) Prove that $p : \tilde{X} \to X$ is "universal" in the sense that, given any covering space $p' : X' \to X$, there is a covering space $q : \tilde{X} \to X'$ with $p = p'q$.

8. (a) Prove that if X is a topological manifold of dimension n then, for each $x_0 \in X$, the relative homology $H_n(X, X - \{x_0\})$ is isomorphic to .

(b) Explain how if X is a C^∞ n-manifold that is orientable, then an orientation of X picks out a particular generator of $H_n(X, X - \{x_0\})$ for each $x_0 \in X$.

9. Let X be the space obtained by deleting from the closed ball of radius 2 in ^{3} the unit circle in the (x, y) plane, i.e.,

$$X = \{(x, y, z) \in {}^3 : x^2 + y^2 + z^2 \leq 2\} \backslash \{(x, y, 0) \in {}^3 : x^2 + y^2 = 1\}.$$

Compute the homology groups of X.

10. Viewing the unit circle S^1 in the plane as the complex numbers of norm one, let $\mu : S^1 \times S^1 \to S^1$ be complex multiplication. Given maps $f, g : S^1 \to S^1$, **define** their "product" $h : S^1 \to S^1$ by $h(z) = \mu(f(z), g(z))$. Prove that the degrees are related by $\deg(h) = \deg(f) + \deg(g)$. (Hint: First show that, for maps of the circle, degree can be defined in terms of the fundamental group.)