Geometry/Topology
QUALIFYING EXAM
Winter 2005

Manifold Problems

1. Let M^2 be a smooth 2-manifold and $f : M^2 \to \mathbf{R}$ be a smooth surjective map without critical points. Assume that for any finite closed interval $[a, b] \hookrightarrow \mathbf{R}$, $f^{-1}([a, b])$ is compact. What is M^2?

2. Show that $T^2 \times S^2$ is parallelizable, i.e., there are 4 vector fields that are everywhere linearly independent.

3. Let $V = P\dfrac{\partial}{\partial x} + Q\dfrac{\partial}{\partial y} + R\dfrac{\partial}{\partial z}$ be a nowhere zero C^∞ vector field on $\mathbf{R}^3$. Show that the following three statements are equivalent.

   a) The orthogonal-to-V plane field is integrable on some neighbourhood of $\mathbf{0} \in \mathbf{R}^3$.

   b) There exists a nowhere-zero C^∞ function $f : \mathbf{R}^3 \to \mathbf{R}$ such that $\mathrm{curl}(fV) \equiv \mathbf{0}$ on some neighbourhood of $\mathbf{0} \in \mathbf{R}^3$.

   c) $V \cdot \mathrm{curl}(V) \equiv 0$ on some neighbourhood of $\mathbf{0} \in \mathbf{R}^3$.

4. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a smooth function and $x \in \mathbf{R}^n$ be a critical point of f. The Hessian $H(t)_x$ at x be a bilinear form: $T_x\mathbf{R}^n \times T_x\mathbf{R}^n \to \mathbf{R}$ defined as follows. For any two vectors V_1 and V_2 in $T_x\mathbf{R}^n$, extend V_2 to a vector field $\tilde{V}_2$ near x, and define $H(f)_x(V_1, V_2) =: D_{v_1}(D_{\tilde{v}_2} f)$.
   Show that:

   (1) $H(f)_x(V_1, V_2) = H(f)_x(V_2, V_1)$.

   (2) $H(f)_x(V_1, V_2)$ is independent of the choice of the extension $\tilde{V}_2$.

5. (1) State Stokes' Theorem in its most general form.
   (2) Use the Stokes' Theorem to prove that for any vector field X defined on $\mathbf{R}^n$, $\int_\Omega (\mathrm{div}\ X) dx^1 \cdots dx^n = \pm \int_{\partial\Omega} (X \cdot N) ds$ where Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$ and a unit normal field N on $\partial\Omega$. Here ds is the "area" form.

Topology Problems

1. Sketch the proof of:

   THEOREM. *If D is a subspace of S^n homeomorphic to I^k for some $k \geq 0$ then the reduced homology groups $\tilde{H}_i(S^n - D, \mathbf{Z})$ are trivial for all i.*

   (Hint: Induction on k.) (This is a special case of Alexander duality. No credit for saying "Applying Alexander duality ...".)

2. Show that $\mathbf{R}P^3$ is not homotopy equivalent to $\mathbf{R}P^2 \vee S^3$. (You could use cup products, degree, or covering spaces.)

3. Suppose $F : X \times I \to Y$ is a homotopy between $f : X \to Y$ and $g : X \to Y$.
   (6 pts) a) Indicate how to construct prism operators $P : C_n(X) \to C_{n+1}(Y)$ satisfying $g_* - f_* = \partial P + P\partial$ where $f_* : C_n(X) \to C_n(Y)$, $g_* : C_n(X) \to C_n(Y)$ are the chain maps.
   (4 pts) b) Show that the induced homomorphisms $H_n(f)$, $H_n(g)$ are equal.

4. Give examples of a) two nonhomeomorphic connected regular 3-sheeted covering spaces of the bouquet of two circles and b) an irregular connected 3-sheeted cover of the bouquet of two circles.

5. (5 pts) a) Find the Euler characteristic of X_4^2, the 2-skeleton of the 4-simplex.
   (5 pts) b) Give a reason why $H_2(X_4^2)$ is free abelian and find its rank.